Quantizations of generalized Cartan type S Lie algebras and of the special algebra $\mathbf{S}(n;\underline{1})$ in the modular case*

Naihong Hu* and Xiuling Wang*

Abstract. The generalized Cartan type **S** Lie algebras in char 0 with the Lie bialgebra structures involved are quantized, where the Drinfel'd twist we used is proved to be a variation of the Jordanian twist. As the passage from char 0 to char p, their quantization integral forms are given. By the modular reduction and base changes, we obtain certain quantizations of the restricted universal enveloping algebra $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ (for the Cartan type simple modular restricted Lie algebra $\mathbf{S}(n;\underline{1})$ of **S** type). They are new Hopf algebras of truncated p-polynomial noncommutative and noncocommutative deformation of dimension $p^{1+(n-1)(p^n-1)}$, which contain the well-known Radford algebra ([**18**]) as a Hopf subalgebra. As a by-product, we also get some Jordanian quantizations for $\mathfrak{sl}_n$, which are induced from those horizontal quantizations of $\mathbf{S}(n;\underline{1})$.

In Hopf algebra or quantum group theory, two standard methods to construct new bialgebras from old ones are by twisting the product by a 2-cocycle but keeping the coproduct unchanged, and by twisting the coproduct by a Drinfel'd twist but preserving the product. Constructing quantizations of Lie bialgebras is an important approach to producing new quantum groups (see [**4, 8**] and references therein). In papers [**6, 7**], Etingof-Kazhdan showed the existence of a universal quantization for Lie bialgebras by constructing a quantization functor. Enriquez-Halbout showed that any coboundary Lie bialgebra, in principle, can be quantized via a certain Etingof-Kazhdan quantization functor (see [**5**]). The Lie bialgebras they considered (including finite- and infinite-dimensional ones) are those classes of the Lie algebras defined by generalized Cartan matrices. However, for another important class of the Cartan type Lie algebras defined by differential operators, they are lack of adequate attention in the literature. In 2004, Grunspan [**11**] obtained the quantization of the (infinite-dimensional) Witt algebra W in characteristic 0 using the twist found by Giaquinto-Zhang ([**9**]), but his way didn't work for the quantum version ([**11**]) of its simple modular Witt algebra $\mathbf{W}(1;\underline{1})$ in characteristic p. In 2005, Song-Su ([**20**]) determined some coboundary triangular Lie bialgebra

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structures (for the definition, see p. 28, [**8**]) on the Lie algebras of the generalized Witt type. The authors ([**12**]) obtained the quantizations both for the generalized-Witt algebra $\mathbf{W}$ in characteristic 0 and for the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$ in characteristic p, where in the rank 1 case, we recovered the Grunspan's work in characteristic 0, and gave the required quantum version in characteristic p.

In the present paper, we continue to treat with the same questions both for the generalized Cartan type $\mathbf{S}$ Lie algebras in characteristic 0 (for the definition, see [**3**]) and for the restricted simple special algebras $\mathbf{S}(n;\underline{1})$ in the modular case (for the definition, see [**22**], [**23**]).

As we known, the construction of Drinfeld twists is difficult. Only a few of twists in explicit forms have been known for a long time (see [**9, 13, 17, 19**] etc.). In this paper, we start an explicit Drinfel'd twist due to [**9**] and [**11**], which, we found recently, is essentially a variation of the Jordanian twist used by Kulish et al (see [**13**], etc.). For this fact, we provide a strict proof in Remark 1.10. Similar to [**11**] and [**12**], we quantize the triangular Lie bialgebra structures on the generalized Cartan type $\mathbf{S}$ Lie algebra in characteristic 0 (The existence of triangular Lie bialgebra structures on it is due to a result of [**15**]). The process depends on the construction of Drinfel'd twists which, up to integral scalars, are controlled by the classical Yang-Baxter r-matrix. To study the modular case, what we discuss first is about the arithmetic property of quantizations to work out their quantization integral forms. To this end, we have to work over the so-called *"positive"* part subalgebra $\mathbf{S}^+$ of the generalized Cartan type $\mathbf{S}$ Lie (shifted) algebra $x^\eta\mathbf{S}$ (where $\eta=-\underline{1}$). This is one of the crucial technical points here. It is an infinite-dimensional simple Lie algebra when defined over a field of characteristic 0, while, defined over a field of characteristic p, it contains a maximal ideal $J_{\underline{1}}$ and the corresponding quotient is exactly the algebra $\mathbf{S}'(n;\underline{1})$. Its derived subalgebra $\mathbf{S}(n;\underline{1})=\mathbf{S}'(n;\underline{1})^{(1)}$ is a Cartan type restricted simple modular Lie algebra of $\mathbf{S}$ type. Secondly, in order to yield the *expected* finite-dimensional quantizations of the restricted universal enveloping algebra of the special algebra $\mathbf{S}(n;\underline{1})$, we need to carry out the modular reduction process: *modulo p reduction* and *modulo "p-restrictedness" reduction*, among which, we have to take the suitable *base changes*. These are the other two crucial technical points. Thirdly, in the process of giving the quantization integral forms for the $\mathbb{Z}$-form $\mathbf{S}^+_{\mathbb{Z}}$ in characteristic 0, we find that there exist $n(n-1)$ the so-called *basic Drinfel'd twists*, which can afford many more Drinfel'd twists (Corollary 5.2). Furthermore, we investigate the twisted structures arisen by these twists. Note that these Hopf algebras contain the well-known Radford algebra ([**18**]) as a Hopf subalgebra. Our work gets a new class of noncommutative and noncocommutative finite-dimensional Hopf algebras in characteristic p (see [**24**]).

The article is organized as follows: In Section 1, we collect some definitions and lemmas which are useful for later use. In Section 2, we quantize explicitly Lie bialgebra structures of generalized Cartan type $\mathbf{S}$ Lie algebra $x^\eta\mathbf{S}$ by the *basic Drinfeld twists* (*in vertical*), and obtain $n(n-1)$ new quantization integral forms for $\mathbf{S}^+_{\mathbb{Z}}$ in characteristic 0. We use this fact to equip the restricted universal enveloping algebra of the special algebra $\mathbf{S}(n;\underline{1})$ with noncommutative and noncocommutative Hopf algebra structures by the modular reduction and the base changes in Section 3. In Section 4, considering some products of pointwise different basic Drinfel'd twists, we can get new quantization integral forms for $\mathbf{S}^+_{\mathbb{Z}}$ in characteristic 0, which, via modulo p reduction and "p-restrictedness" reduction, together with two steps of

base changes, eventually leads to new Hopf algebras of dimension $p^{1+(n-1)(p^n-1)}$ with indeterminate t or of dimension $p^{(n-1)(p^n-1)}$ with specializing t into a scalar in $\mathcal{K}$ in characteristic p. In Section 5, using the *horizontal* twists (different from the *vertical* ones in Section 3), we get some new quantizations of horizontal type of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$, which contain some quantizations of the Lie algebra $\mathfrak{sl}_n$ derived by the Jordanian twists (cf. [**14**]).

1. Preliminaries

1.1. Generalized Cartan type S Lie algebra and its Lie bialgebra structure. Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F})=0$ and $n>0$. Let $\mathbb{Q}_n=\mathbb{F}[x_1^{\pm1},\cdots,x_n^{\pm1}]$ be a Laurent polynomial algebra and ∂_i coincides with the degree operator $x_i\frac{\partial}{\partial x_i}$. Set $T=\bigoplus_{i=1}^n\mathbb{Z}\partial_i$, and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^n$.

Denote $\mathbf{W}=\mathbb{Q}_n\otimes_{\mathbb{Z}}T=\text{Span}_{\mathbb{F}}\{\,x^\alpha\partial\mid\alpha\in\mathbb{Z}^n,\partial\in T\}$, where we set $x^\alpha\partial=x^\alpha\otimes\partial$ for short. Then $\mathbf{W}=\text{Der}_{\mathbb{F}}(\mathbb{Q}_n)$ is a Lie algebra of generalized-Witt type (see [**2**]) under the following bracket

$$[\,x^\alpha\partial,\,x^\beta\partial'\,]=x^{\alpha+\beta}\big(\partial(\beta)\partial'-\partial'(\alpha)\partial\big),\qquad\forall\;\alpha,\,\beta\in\mathbb{Z}^n;\;\partial,\,\partial'\in T,$$

where $\partial(\beta)=\langle\partial,\beta\rangle=\langle\beta,\partial\rangle=\sum\limits_{i=1}^n a_i\beta_i\in\mathbb{Z}$ for $\partial=\sum\limits_{i=1}^n a_i\partial_i\in T$ and $\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{Z}^n$. The bilinear map $\langle\cdot,\cdot\rangle:T\times\mathbb{Z}^n\longrightarrow\mathbb{Z}$ is non-degenerate in the sense that

$$\partial(\alpha)=\langle\partial,\alpha\rangle=0\quad(\forall\;\partial\in T),\;\Longrightarrow\alpha=0,$$
$$\partial(\alpha)=\langle\partial,\alpha\rangle=0\quad(\forall\;\alpha\in\mathbb{Z}^n),\;\Longrightarrow\partial=0.$$

$\mathbf{W}$ is an infinite dimensional simple Lie algebra over $\mathbb{F}$ (see [**2**]).

We recall that the *divergence* (cf. [**3**]) div : $\mathbf{W}\longrightarrow\mathbb{Q}_n$ is the $\mathbb{F}$-linear map such that

$$\text{div}(x^\alpha\partial)=\partial(x^\alpha)=\partial(\alpha)x^\alpha,\quad\textit{for }\alpha\in\mathbb{Z}^n,\;\partial\in T.\tag{1}$$

The *divergence* has the following two properties:

$$\text{div}([u,v])=u\cdot\text{div}(v)-v\cdot\text{div}(u),\tag{2}$$
$$\text{div}(fw)=f\,\text{div}(w)+w\cdot f,\tag{3}$$

for $u,v,w\in\mathbf{W},f\in\mathbb{Q}_n$. In view of (2), the subspace

$$\widetilde{\mathbf{S}}=\text{Ker}(\text{div})$$

is a subalgebra of $\mathbf{W}$.

The Lie algebra $\mathbf{W}$ is $\mathbb{Z}^n$-graded, whose homogeneous components are

$$\mathbf{W}_\alpha:=x^\alpha T,\qquad\alpha\in\mathbb{Z}^n.$$

The *divergence* div : $\mathbf{W}\longrightarrow\mathbb{Q}_n$ is a derivation of degree 0. Hence, its kernel is a homogeneous subalgebra of $\mathbf{W}$. So we have

$$\widetilde{\mathbf{S}}=\bigoplus_{\alpha\in\mathbb{Z}^n}\widetilde{\mathbf{S}}_\alpha,\quad\widetilde{\mathbf{S}}_\alpha:=\widetilde{\mathbf{S}}\cap\mathbf{W}_\alpha.$$

For each $\alpha\in\mathbb{Z}^n$, let $\hat{\alpha}:T\to\mathbb{F}$ be the corresponding linear function defined by $\hat{\alpha}(\partial)=\langle\partial,\alpha\rangle=\partial(\alpha)$. We have

$$\widetilde{\mathbf{S}}_\alpha=x^\alpha T_\alpha,\quad\textit{and}\;\;T_\alpha=\text{Ker}(\hat{\alpha}).$$

The algebra $\widetilde{\mathbf{S}}$ is not simple, but its derived subalgebra $\mathbf{S} = (\widetilde{\mathbf{S}})'$ is simple, assuming only that $\dim T \geq 2$. According to Proposition 3.1 [**3**], we have $\mathbf{S} = \bigoplus_{\alpha \neq 0} \widetilde{\mathbf{S}}_\alpha$. More generally, it turns out that the shifted spaces $x^\eta \mathbf{S}$, $\eta \in \mathbb{Z}^n - \{0\}$, are simple subalgebras of $\mathbf{W}$ if $\dim T \geq 3$. We refer to the simple Lie algebras $x^\eta \mathbf{S}$ as the Lie algebras of *generalized Cartan type* $\mathbf{S}$ (see [**3**]). The Lie algebra $x^\eta \mathbf{S}$ is $\mathbb{Z}^n$-graded with $x^\alpha T_{\alpha-\eta}$ $(\alpha \neq \eta)$, as its homogeneous component of degree α, while its homogeneous component of degree η is 0.

Throughout this paper, we assume that $\eta \neq 0$, $\eta_k = \eta_{k'}$.

Take two distinguished elements $h = \partial_k - \partial_{k'}, e = x^\gamma \partial_0 \in x^\eta \mathbf{S}$ such that $[h, e] = e$ where $1 \leq k \neq k' \leq n$. It is easy to see that $\partial_0(\gamma - \eta) = 0$, and $\gamma_k - \gamma_{k'} = 1$. Using a result of [**15**], we have the following

Proposition 1.1. *There is a triangular Lie bialgebra structure on $x^\eta \mathbf{S}$ $(\eta \neq 0,\ \eta_k = \eta_{k'})$ given by the classical Yang-Baxter r-matrix*

$$r := (\partial_k - \partial_{k'}) \otimes x^\gamma \partial_0 - x^\gamma \partial_0 \otimes (\partial_k - \partial_{k'}), \quad \forall\, \partial_{k'},\, \partial_k \in T,\ \gamma \in \mathbb{Z}^n,$$

where $\gamma_k - \gamma_{k'} = 1$, $\partial_0(\gamma) = \partial_0(\eta)$ and $[\,\partial_k - \partial_{k'}, x^\gamma \partial_0\,] = x^\gamma \partial_0$. $\square$

1.2. Generalized Cartan type S Lie subalgebra $\mathbf{S}^+$. Denote $D_i = \frac{\partial}{\partial x_i}$. Set $\mathbf{W}^+ := \mathrm{Span}_\mathcal{K}\{x^\alpha D_i \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Then $\mathbf{W}^+ = \mathrm{Der}_\mathcal{K}(\mathcal{K}[x_1, \cdots, x_n])$ is the derivation Lie algebra of polynomial ring $\mathcal{K}[x_1, \cdots, x_n]$, which, via the identification $x^\alpha D_i$ with $x^{\alpha-\epsilon_i}\partial_i$ (here $\alpha - \epsilon_i \in \mathbb{Z}^n$, $\epsilon_i = (\delta_{1i}, \cdots, \delta_{ni})$), can be viewed as a Lie subalgebra (the "positive" part) of the generalized-Witt algebra $\mathbf{W}$ over a field $\mathcal{K}$.

For $X = \sum_{i=1}^n a_i D_i \in \mathbf{W}$, we define $\mathrm{Div}(X) = \sum_{i=1}^n D_i(a_i)$ as usual. Note that $\mathrm{div}(X) = \sum_{i=1}^n x_i D_i(x_i^{-1} a_i)$ (since $\partial_i = x_i D_i$). Thus we have $\mathrm{div}(x_1 \cdots x_n X) = x_1 \cdots x_n \mathrm{Div}(X)$. This means that $X \in \mathrm{Ker}(\mathrm{Div})$ if and only if $x^{\underline{1}} X \in \widetilde{\mathbf{S}}$, and if and only if $X \in x^{-\underline{1}}\widetilde{\mathbf{S}}$, where $\underline{1} = \epsilon_1 + \cdots + \epsilon_n$.

Set $\mathbf{S}^+ := \mathrm{Ker}(\mathrm{Div}) \bigcap \mathbf{W}^+$, then we have $\mathbf{S}^+ = (x^{-\underline{1}}\mathbf{S}) \bigcap \mathbf{W}^+$ since $\mathbf{S} = \bigoplus_{\alpha \neq 0} \widetilde{\mathbf{S}}_\alpha$ and $x^{-\underline{1}}\widetilde{\mathbf{S}}_0 \bigcap \mathbf{W}^+ = 0$ (where $\widetilde{\mathbf{S}}_0 = T$), which is a subalgebra of $\mathbf{W}^+$. Note that $\{\, \alpha_n x^{\alpha-\epsilon_n} D_i - \alpha_i x^{\alpha-\epsilon_i} D_n \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i < n \,\}$ is a basis of $\mathbf{S}^+$, where $\alpha_n x^{\alpha-\epsilon_n} D_i - \alpha_i x^{\alpha-\epsilon_i} D_n = x^{\alpha-\epsilon_i-\epsilon_n}(\alpha_n \partial_i - \alpha_i \partial_n) \in x^{\alpha-\epsilon_i-\epsilon_n} T_{\alpha-\epsilon_i-\epsilon_n+\underline{1}}$ indicates once again that $\mathbf{S}^+$ is indeed a subalgebra of $x^{-\underline{1}}\mathbf{S}$ since $\partial_i = x_i D_i$.

1.3. The special algebra $\mathbf{S}(n;\underline{1})$. Assume now that $\mathrm{char}(\mathcal{K}) = p$, then by definition, the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$ is a restricted simple Lie algebra over a field $\mathcal{K}$. Its structure of p-Lie algebra is given by $D^{[p]} = D^p$, $\forall\, D \in \mathbf{W}(n;\underline{1})$ with a basis $\{\, x^{(\alpha)} D_j \mid 1 \leq j \leq n,\ 0 \leq \alpha \leq \tau \}$, where $\tau = (p-1, \cdots, p-1) \in \mathbb{N}^n$; $\epsilon_i = (\delta_{1i}, \cdots, \delta_{ni})$ such that $x^{(\epsilon_i)} = x_i$ and $D_j(x_i) = \delta_{ij}$; and $\mathcal{O}(n;\underline{1}) := \{\, x^{(\alpha)} \mid 0 \leq \alpha \leq \tau\}$ is the restricted divided power algebra with $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$ and a convention: $x^{(\alpha)} = 0$ if α has a component $\alpha_j < 0$ or $\geq p$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^n \binom{\alpha_i+\beta_i}{\alpha_i}$. Note that $\mathcal{O}(n;\underline{1})$ is $\mathbb{Z}$-graded by $\mathcal{O}(n;\underline{1})_i := \mathrm{Span}_\mathcal{K}\{\, x^{(\alpha)} \mid 0 \leq \alpha \leq \tau, |\alpha| = i\}$, where $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, $\mathbf{W}(n;\underline{1})$ is isomorphic to $\mathrm{Der}_\mathcal{K}(\mathcal{O}(n;\underline{1}))$ and inherits a gradation from $\mathcal{O}(n;\underline{1})$ by means of $\mathbf{W}(n;\underline{1})_i = \sum_{j=1}^n \mathcal{O}(n;\underline{1})_{i+1} D_j$. Then the subspace

$$\mathbf{S}'(n;\underline{1}) = \{E \in \mathbf{W}(n;\underline{1}) \mid \mathrm{Div}(E) = 0\}$$

is a subalgebra of $\mathbf{W}(n;\underline{1})$.

Its derived subalgebra $\mathbf{S}(n;\underline{1}) = \mathbf{S}'(n;\underline{1})^{(1)}$ is called *the special algebra*. Then $\mathbf{S}(n;\underline{1}) = \bigoplus_{i=-1}^s \mathbf{S}(n;\underline{1}) \cap \mathbf{W}(n;\underline{1})_i$ is graded with $s = |\tau| - 2$. Recall the mappings

$D_{ij}: \mathcal{O}(n;\underline{1}) \longrightarrow \mathbf{W}(n;\underline{1})$, $D_{ij}(f) = D_j(f)D_i - D_i(f)D_j$ for $f \in \mathcal{O}(n;\underline{1})$. Note that $D_{ii} = 0$ and $D_{ij} = -D_{ji}$, $1 \le i, j \le n$. Then by Lemma 4.2.2 [**22**],

$$\mathbf{S}(n;\underline{1}) = \mathrm{Span}_{\mathcal{K}}\{D_{in}(f) \mid f \in \mathcal{O}(n;\underline{1}), 1 \le i < n\}$$

is a p-subalgebra of $\mathbf{W}(n;\underline{1})$ with restricted gradation. Evidently, we have the following result (see the proof of Theorem 3.7, p.159 in [**23**])

LEMMA 1.2. $\mathbf{S}'(n;\underline{1}) = \mathbf{S}(n;\underline{1}) + \sum\limits_{j=1}^{n} \mathcal{K}x^{(\tau-(p-1)\epsilon_j)}D_j$. *And* $\dim_{\mathcal{K}}\mathbf{S}'(n;\underline{1}) = (n-1)p^n + 1$, $\dim_{\mathcal{K}}\mathbf{S}(n;\underline{1}) = (n-1)(p^n-1)$. □

By definition (cf. [**22**]), the restricted universal enveloping algebra $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ is isomorphic to $U(\mathbf{S}(n;\underline{1}))/I$ where I is the Hopf ideal of $U(\mathbf{S}(n;\underline{1}))$ generated by $(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})$, $(D_{ij}(x^{(\alpha)}))^p$ with $\alpha \neq \epsilon_i + \epsilon_j$ for $1 \le i < j \le n$. Since $\dim_{\mathcal{K}}\mathbf{S}(n;\underline{1}) = (n-1)(p^n-1)$, we have $\dim_{\mathcal{K}}\mathbf{u}(\mathbf{S}(n;\underline{1})) = p^{(n-1)(p^n-1)}$.

1.4. A crucial Lemma. For any element x of a unital R-algebra (R a ring) and $a \in R$, we set (see [**9**])

$$x_a^{\langle n\rangle} := (x+a)(x+a+1)\cdots(x+a+n-1), \tag{4}$$

then $x^{\langle n\rangle} := x_0^{\langle n\rangle} = \sum_{k=0}^n c(n,k)x^k$ where $c(n,k)$ is the number of $\pi \in \mathfrak{S}_n$ with exactly k cycles (cf. [**21**]). Given a $\pi \in \mathfrak{S}_n$, let $c_i = c_i(\pi)$ be the number of cycles of π of length i. Note that $n = \sum ic_i$. Define the type of π, denoted type π, to be the n-tuple $\underline{c} = (c_1, \cdots, c_n)$. The total number of cycles of π is denoted $c(\pi)$, so $c(\pi) = |\,\underline{c}\,| = c_1 + \cdots + c_n$. Denote by $\mathfrak{S}_n(\underline{c})$ the set of all $\sigma \in \mathfrak{S}_n$ of type $\underline{c}$, then $|\mathfrak{S}_n(\underline{c})| = n!/1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!$ (see Proposition 1.3.2 [**21**]).

We also set

$$x_a^{[n]} := (x+a)(x+a-1)\cdots(x+a-n+1), \tag{5}$$

then $x^{[n]} := x_0^{[n]} = \sum_{k=0}^n s(n,k)x^k$ where $s(n,k) = (-1)^{n-k}c(n,k)$ is the Stirling number of the first kind.

LEMMA 1.3. ([**9, 11**]) *For any element x of a unital $\mathbb{F}$-algebra with char$(\mathbb{F}) = 0$, a, $b \in \mathbb{F}$ and r, s, $t \in \mathbb{Z}$, one has*

$$x_a^{\langle s+t\rangle} = x_a^{\langle s\rangle}\, x_{a+s}^{\langle t\rangle}, \tag{6}$$

$$x_a^{[s+t]} = x_a^{[s]}\, x_{a-s}^{[t]}, \tag{7}$$

$$x_a^{[s]} = x_{a-s+1}^{\langle s\rangle}, \tag{8}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!\,t!}\, x_a^{[s]}\, x_b^{\langle t\rangle} = \binom{a-b}{r} = \frac{(a-b)\cdots(a-b-r+1)}{r!}, \tag{9}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!\,t!}\, x_a^{[s]}\, x_{b-s}^{[t]} = \binom{a-b+r-1}{r} = \frac{(a-b)\cdots(a-b+r-1)}{r!}. \tag{10}$$

1.5. Quantization by Drinfel'd twists. The following result is well-known (see [**1, 4, 8, 19**], etc.).

LEMMA 1.4. *Let $(A, m, \iota, \Delta_0, \varepsilon_0, S_0)$ be a Hopf algebra over a commutative ring. A Drinfel'd twist $\mathcal{F}$ on A is an invertible element of $A \otimes A$ such that*

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\mathrm{Id} \otimes \Delta_0)(\mathcal{F}),$$
$$(\varepsilon_0 \otimes \mathrm{Id})(\mathcal{F}) = 1 = (\mathrm{Id} \otimes \varepsilon_0)(\mathcal{F}).$$

Then, $w = m(\mathrm{Id}\otimes S_0)(\mathcal{F})$ *is invertible in* A *with* $w^{-1} = m(S_0\otimes \mathrm{Id})(\mathcal{F}^{-1})$.

Moreover, if we define $\Delta : A \longrightarrow A\otimes A$ *and* $S : A \longrightarrow A$ *by*

$$\Delta(a) = \mathcal{F}\Delta_0(a)\mathcal{F}^{-1}, \qquad S = w\, S_0(a)\, w^{-1},$$

then $(A, m, \iota, \Delta, \varepsilon, S)$ *is a new Hopf algebra, called the twisting of* A *by the Drinfel'd twist* $\mathcal{F}$.

Let $\mathbb{F}[[t]]$ be a ring of formal power series over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Assume that L is a triangular Lie bialgebra over $\mathbb{F}$ with a classical Yang-Baxter r-matrix r (see [**4, 8**]). Let $U(L)$ denote the universal enveloping algebra of L, with the standard Hopf algebra structure $(U(L), m, \iota, \Delta_0, \varepsilon_0, S_0)$.

Let us consider *the topologically free* $\mathbb{F}[[t]]$*-algebra* $U(L)[[t]]$ (for the definition, see p. 4, [**8**]), which can be viewed as an associative $\mathbb{F}$-algebra of formal power series with coefficients in $U(L)$. Naturally, $U(L)[[t]]$ equips with an induced Hopf algebra structure arising from that on $U(L)$ (via the coefficient ring extension), by abuse of notation, denoted still by $(U(L)[[t]], m, \iota, \Delta_0, \varepsilon_0, S_0)$.

Definition 1.5. ([**12**]) For a triangular Lie bialgebra L over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$, $U(L)[[t]]$ is called *a quantization of* $U(L)$ *by a Drinfel'd twist* $\mathcal{F}$ over $U(L)[[t]]$ if $U(L)[[t]]/tU(L)[[t]] \cong U(L)$, and $\mathcal{F}$ is determined by its r-matrix r (namely, its Lie bialgebra structure).

1.6. Construction of Drinfel'd twists. Let L be a Lie algebra containing linearly independent elements h and e satisfying $[h,\, e] = e$, then the classical Yang-Baxter r-matrix $r = h\otimes e - e\otimes h$ equips L with the structure of triangular coboundary Lie bialgebra (see [**15**]). To describe a quantization of $U(L)$ by a Drinfel'd twist $\mathcal{F}$ over $U(L)[[t]]$, we need an explicit construction for such a Drinfel'd twist. In what follows, we shall see that such a twist depends upon the choice of *two distinguished elements* h, e arising from its r-matrix r.

Recall the following results proved in [**11**] and [**12**]. Note that h and e satisfy the following equalities:

$$e^s\cdot h_a^{[m]} = h_{a-s}^{[m]}\cdot e^s, \tag{11}$$

$$e^s\cdot h_a^{\langle m\rangle} = h_{a-s}^{\langle m\rangle}\cdot e^s, \tag{12}$$

where m, s are non-negative integers, $a\in\mathbb{F}$.

For $a\in\mathbb{F}$, following [**11**], we set

$$\mathcal{F}_a = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_a^{[r]}\otimes e^r t^r, \qquad F_a = \sum_{r=0}^{\infty}\frac{1}{r!}h_a^{\langle r\rangle}\otimes e^r t^r,$$
$$u_a = m\cdot(S_0\otimes\mathrm{Id})(F_a), \qquad v_a = m\cdot(\mathrm{Id}\otimes S_0)(\mathcal{F}_a).$$

Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$.

Since $S_0(h_a^{\langle r\rangle}) = (-1)^r h_{-a}^{[r]}$ and $S_0(e^r) = (-1)^r e^r$, one has

$$v_a = \sum_{r=0}^{\infty}\frac{1}{r!}h_a^{[r]}e^r t^r, \quad u_b = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-b}^{[r]}e^r t^r.$$

Lemma 1.6. ([**11**]) *For* a, $b\in\mathbb{F}$, *one has*

$$\mathcal{F}_aF_b = 1\otimes(1-et)^{a-b} \quad \text{and} \quad v_au_b = (1-et)^{-(a+b)}.$$

Corollary 1.7. ([**11**]) For $a \in \mathbb{F}$, $\mathcal{F}_a$ and u_a are invertible with $\mathcal{F}_a^{-1} = F_a$ and $u_a^{-1} = v_{-a}$. In particular, $\mathcal{F}^{-1} = F$ and $u^{-1} = v$.

Lemma 1.8. ([**11**]) *For any positive integers r, we have*

$$\Delta_0(h^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h^{[i]} \otimes h^{[r-i]}.$$

Furthermore, $\Delta_0(h^{[r]}) = \sum\limits_{i=0}^{r} \binom{r}{i} h_{-s}^{[i]} \otimes h_s^{[r-i]}$ *for any $s \in \mathbb{F}$.*

Proposition 1.9. ([**11**]) *If a Lie algebra L contains a 2-dimensional solvable Lie subalgebra with a basis $\{h, e\}$ satisfying $[h, e] = e$, then $\mathcal{F} = \sum\limits_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]} \otimes e^r t^r$ is a Drinfel'd twist on $U(L)[[t]]$.*

Remark 1.10. Recently, we observed that Kulish et al earlier used the so-called *Jordanian twist* (see [**13**]) with the two-dimensional carrier subalgebra $B(2)$ such that $[H, E] = E$, defined by the canonical twisting element

$$\mathcal{F}_{\mathcal{J}}^c = \exp(H \otimes \sigma(t)), \quad \sigma(t) = \ln(1 + Et),$$

where $\exp(X) = \sum_{i=0}^{\infty} \frac{X^n}{n!}$ and $\ln(1 + X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n$.

Expanding it, we get

$$\begin{aligned}
\exp(H \otimes \sigma(t)) &= \exp\Big(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} H \otimes (Et)^n\Big) \\
&= \prod_{n\geq 1} \Big(\sum_{\ell \geq 0} \frac{(-1)^{(n+1)\ell}}{n^\ell \ell!} H^\ell \otimes (Et)^{n\ell}\Big) \\
&= \sum_{n \geq 1} \sum_{c_1, \cdots, c_n \geq 0} \frac{(-1)^{c_1 + 2c_2 + \cdots + nc_n - |\,\underline{c}\,|}}{c_1! \cdots c_n! 1^{c_1} 2^{c_2} \cdots n^{c_n}} H^{|\,\underline{c}\,|} \otimes (Et)^{c_1 + 2c_2 + \cdots + nc_n} \\
&= \sum_{n \geq 0} \Big(\sum_{\underline{c}} \frac{(-1)^{n - |\,\underline{c}\,|} |\, \mathfrak{S}_n(\underline{c}) \,|}{n!} H^{|\,\underline{c}\,|}\Big) \otimes (Et)^n \\
&= \sum_{n \geq 0} \left(\sum_{k=0}^{n} \frac{(-1)^{n-k} c(n,k)}{n!} H^k\right) \otimes (Et)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} H^{[n]} \otimes E^n t^n,
\end{aligned}$$

where we set $n = c_1 + 2c_2 + \cdots + nc_n$, $c(n,k) = \sum_{|\,\underline{c}\,| = k} |\, \mathfrak{S}_n(\underline{c}) \,|$. So

$$(\mathcal{F}_{\mathcal{J}}^c)^{-1} = \exp((-H) \otimes \sigma(t)) = \sum_{r=0}^{\infty} \frac{1}{r!} (-H)^{[r]} \otimes E^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} H^{\langle r \rangle} \otimes E^r t^r.$$

Consequently, we can rewrite the twist $\mathcal{F}$ in Proposition 1.9 as

$$\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} H^{[r]} \otimes E^r t^r = \exp(H \otimes \sigma'(t)), \quad \sigma'(t) = \ln(1 - Et),$$

where $[H, -E] = -E$. So there is no difference between the twists $\mathcal{F}$ and $\mathcal{F}_{\mathcal{J}}^c$. They are essentially the same up to an isomorphism on the carrier subalgebra $B(2)$.

2. Quantization of Lie bialgebra of generalized Cartan type S

In this section, we explicitly quantize the Lie bialgebras $x^\eta\mathbf{S}$ of generalized Cartan type $\mathbf{S}$ by the twist given in Proposition 1.9.

2.1. Some commutative relations in $U(x^\eta\mathbf{S})$. For the universal enveloping algebra $U(x^\eta\mathbf{S})$ of the generalized Cartan type $\mathbf{S}$ Lie algebra $x^\eta\mathbf{S}$ over $\mathbb{F}$, we need to do some necessary calculations, which are important to the quantizations of Lie bialgebra structure of $x^\eta\mathbf{S}$ in the sequel.

Lemma 2.1. *Fix two distinguished elements $h := \partial_k - \partial_{k'}$, $e := x^\gamma\partial_0 \in x^\gamma T_{\gamma-\eta}$ with $\gamma_k - \gamma_{k'} = 1$ for $x^\eta\mathbf{S}$. For $a \in \mathbb{F}$, $x^\alpha\partial \in x^\alpha T_{\alpha-\eta}$, $x^\beta\partial' \in x^\beta T_{\beta-\eta}$, m is non-negative integer, the following equalities hold in $U(x^\eta\mathbf{S})$:*

$$x^\alpha\partial \cdot h_a^{[m]} = h_{a+(\alpha_{k'}-\alpha_k)}^{[m]} \cdot x^\alpha\partial, \tag{13}$$

$$x^\alpha\partial \cdot h_a^{\langle m\rangle} = h_{a+(\alpha_{k'}-\alpha_k)}^{\langle m\rangle} \cdot x^\alpha\partial, \tag{14}$$

$$x^\alpha\partial \cdot (x^\beta\partial')^m = \sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}(x^\beta\partial')^{m-\ell} \cdot x^{\alpha+\ell\beta}\Big(a_\ell\partial - b_\ell\partial'\Big), \tag{15}$$

where $a_\ell = \prod_{j=0}^{\ell-1}\partial'(\alpha+j\beta) = \prod_{j=0}^{\ell-1}\partial'(\alpha+j\eta)$, $b_\ell = \ell\,\partial(\beta)a_{\ell-1}$, and set $a_0 = 1$, $b_0 = 0$.

Proof. One has (13) and (14) by using induction on m.

Formula (15) is a consequence of the fact (see Proposition 1.3 (4), [**23**]) that for any elements a, c in an associative algebra, one has

$$c\,a^m = \sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}a^{m-\ell}(\mathrm{ad}\,a)^\ell(c),$$

together with the formula

$$(\mathrm{ad}\,x^\beta\partial')^\ell(x^\alpha\partial) = x^{\alpha+\ell\beta}(a_\ell\partial - b_\ell\partial'), \tag{16}$$

obtained by induction on ℓ when taking $a = x^\beta\partial'$, $c = x^\alpha\partial$. □

To simplify formulas in the sequel, we introduce the operator $d^{(\ell)}(\ell \geq 0)$ on $U(x^\eta\mathbf{S})$ defined by $d^{(\ell)} := \frac{1}{\ell!}(\mathrm{ad}\,e)^\ell$. From (16) and the derivation property of $d^{(\ell)}$, it is easy to get

Lemma 2.2. *For $\mathbb{Z}^n$-homogeneous elements $x^\alpha\partial$, a_i, the following equalities hold in $U(x^\eta\mathbf{S})$:*

$$d^{(\ell)}(x^\alpha\partial) = x^{\alpha+\ell\gamma}(A_\ell\partial - B_\ell\partial_0), \tag{17}$$

$$d^{(\ell)}(a_1\cdots a_s) = \sum_{\ell_1+\cdots+\ell_s=\ell} d^{(\ell_1)}(a_1)\cdots d^{(\ell_s)}(a_s), \tag{18}$$

where $A_\ell = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}\partial_0(\alpha+j\gamma) = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}\partial_0(\alpha+j\eta)$, $B_\ell = \partial(\gamma)A_{\ell-1}$, and set $A_0 = 1$, $A_{-1} = 0$.

Denote by $(U(x^\eta\mathbf{S}), m, \iota, \Delta_0, S_0, \varepsilon_0)$ the standard Hopf algebra structure of the universal enveloping algebra $U(x^\eta\mathbf{S})$ for the Lie algebra $x^\eta\mathbf{S}$.

2.2. Quantization of $U(x^\eta \mathbf{S})$ in char 0. We can perform the process of twisting the standard Hopf structure $(U(x^\eta\mathbf{S})[[t]],\, m,\, \iota,\, \Delta_0,\, S_0,\, \varepsilon_0)$ by the Drinfel'd twist $\mathcal{F}$ constructed in Proposition 1.9.

The following Lemma is very useful to our main result in this section.

LEMMA 2.3. *For $a\in\mathbb{F}$, $\alpha\in\mathbb{Z}^n$, and $x^\alpha\partial\in x^\alpha T_{\alpha-\eta}$, one has*

$$\big((x^\alpha\partial)^s\otimes 1\big)\cdot F_a = F_{a+s(\alpha_{k'}-\alpha_k)}\cdot\big((x^\alpha\partial)^s\otimes 1\big), \tag{19}$$

$$(x^\alpha\partial)^s\cdot u_a = u_{a+s(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty} d^{(\ell)}\big((x^\alpha\partial)^s\big)\cdot h_{1-a}^{\langle\ell\rangle}t^\ell\Big), \tag{20}$$

$$\big(1\otimes(x^\alpha\partial)^s\big)\cdot F_a = \sum_{\ell=0}^{\infty}(-1)^\ell F_{a+\ell}\cdot\Big(h_a^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^s)t^\ell\Big). \tag{21}$$

PROOF. For (19): By (14), one has

$$\begin{aligned}(x^\alpha\partial\otimes 1)\cdot F_a &= \sum_{m=0}^{\infty}\frac{1}{m!}x^\alpha\partial\cdot h_a^{\langle m\rangle}\otimes e^m t^m\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}h^{\langle m\rangle}_{a+(\alpha_{k'}-\alpha_k)}\cdot x^\alpha\partial\otimes e^m t^m\\ &= F_{a+(\alpha_{k'}-\alpha_k)}\cdot(x^\alpha\partial\otimes 1).\end{aligned}$$

By induction on s, we obtain the result.

For (20): Let $a_\ell=\prod_{j=0}^{\ell-1}\partial_0(\alpha+j\gamma)$, $b_\ell=\ell\,\partial(\gamma)a_{\ell-1}$, using induction on s. For $s=1$, using (7), (11), (13) and (15), we get

$$\begin{aligned}x^\alpha\partial\cdot u_a &= x^\alpha\partial\cdot\Big(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a}^{[r]}\cdot e^r t^r\Big)\\ &= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}x^\alpha\partial\cdot h_{-a}^{[r]}\cdot e^r t^r\\ &= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-(\alpha_k-\alpha_{k'})}\cdot x^\alpha\partial\cdot e^r t^r\\ &= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-(\alpha_k-\alpha_{k'})}\Big(\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}e^{r-\ell}\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0)t^r\Big)\\ &= \sum_{r,\ell=0}^{\infty}\frac{(-1)^{r+\ell}}{(r+\ell)!}h^{[r+\ell]}_{-a-(\alpha_k-\alpha_{k'})}\Big((-1)^\ell\binom{r+\ell}{\ell}e^r\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0)t^{r+\ell}\Big)\\ &= \sum_{r,\ell=0}^{\infty}\frac{(-1)^r}{r!\ell!}h^{[r]}_{-a-(\alpha_k-\alpha_{k'})}\cdot h^{[\ell]}_{-a-(\alpha_k-\alpha_{k'})-r}\cdot e^r\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0)t^{r+\ell}\\ &= \sum_{\ell=0}^{\infty}\Big(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-(\alpha_k-\alpha_{k'})}e^r t^r\Big)h^{[\ell]}_{-a-(\alpha_k-\alpha_{k'})}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0)t^\ell\\ &= u_{a+(\alpha_k-\alpha_{k'})}\cdot\sum_{\ell=0}^{\infty}h^{[\ell]}_{-a-(\alpha_k-\alpha_{k'})}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0)t^\ell\end{aligned}$$

$$= u_{a+(\alpha_k-\alpha_{k'})}\cdot\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0)\cdot h^{[\ell]}_{-a+\ell}t^\ell$$

$$= u_{a+(\alpha_k-\alpha_{k'})}\cdot\sum_{\ell=0}^{\infty}d^{(\ell)}(x^\alpha\partial)\cdot h^{\langle\ell\rangle}_{-a+1}t^\ell,$$

where $A_\ell=\frac{1}{\ell!}\prod_{j=0}^{\ell-1}\partial_0(\alpha+j\gamma)=\frac{1}{\ell!}\prod_{j=0}^{\ell-1}\partial_0(\alpha+j\eta)$, $B_\ell=\partial(\gamma)A_{\ell-1}$, and set $A_0=1$, $A_{-1}=0$.

Suppose $s\geq 1$. Using Lemma 2.2 and the induction hypothesis on s, we have

$$\begin{aligned}
(x^\alpha\partial)^{s+1}\cdot u_a &= x^\alpha\partial\cdot u_{a+s(\alpha_k-\alpha_{k'})}\cdot\sum_{n=0}^{\infty}d^{(n)}((x^\alpha\partial)^s)\cdot h^{\langle n\rangle}_{1-a}t^n\\
&= u_{a+(s+1)(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{m=0}^{\infty}d^{(m)}(x^\alpha\partial)\cdot h^{\langle m\rangle}_{1-a-s(\alpha_k-\alpha_{k'})}t^m\Big)\\
&\qquad\cdot\Big(\sum_{n=0}^{\infty}d^{(n)}((x^\alpha\partial)^s)\cdot h^{\langle n\rangle}_{1-a}t^n\Big)\\
&= u_{a+(s+1)(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{m,n=0}^{\infty}d^{(m)}(x^\alpha\partial)d^{(n)}((x^\alpha\partial)^s)h^{\langle m\rangle}_{1-a+n}h^{\langle n\rangle}_{1-a}t^{n+m}\Big)\\
&= u_{a+(s+1)(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}\sum_{m+n=\ell}d^{(m)}(x^\alpha\partial)d^{(n)}((x^\alpha\partial)^s)h^{\langle\ell\rangle}_{1-a}t^\ell\Big)\\
&= u_{a+(s+1)(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}((x^\alpha\partial)^{s+1})h^{\langle\ell\rangle}_{1-a}t^\ell\Big),
\end{aligned}$$

where we get the first and second "=" by using the inductive hypothesis, the third by using (14) & (18) and the fourth by using (6) & (18).

For (21): For s=1, using (15) we get

$$\begin{aligned}
(1\otimes x^\alpha\partial)\cdot F_a &= \sum_{m=0}^{\infty}\frac{1}{m!}h^{\langle m\rangle}_a\otimes x^\alpha\partial\cdot e^m t^m\\
&= \sum_{m=0}^{\infty}\frac{1}{m!}h^{\langle m\rangle}_a\otimes\left(\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}e^{m-\ell}\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0)t^m\right)\\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^\ell\frac{1}{m!\ell!}h^{\langle m+\ell\rangle}_a\otimes e^m\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0)t^{m+\ell}\\
&= \sum_{\ell=0}^{\infty}(-1)^\ell\left(\sum_{m=0}^{\infty}\frac{1}{m!}h^{\langle m\rangle}_{a+\ell}\otimes e^m t^m\right)\Big(h^{\langle\ell\rangle}_a\otimes d^{(\ell)}(x^\alpha\partial)t^\ell\Big)\\
&= \sum_{\ell=0}^{\infty}(-1)^\ell F_{a+\ell}\cdot\Big(h^{\langle\ell\rangle}_a\otimes d^{(\ell)}(x^\alpha\partial)t^\ell\Big).
\end{aligned}$$

For $s>1$, it follows from the induction hypothesis & (18). □

The following theorem gives the quantization of $U(x^\eta\mathbf{S})$ by Drinfel'd twist $\mathcal{F}$, which is essentially determined by the Lie bialgebra triangular structure on $x^\eta\mathbf{S}$.

Theorem 2.4. *Fix two distinguished elements $h=\partial_k-\partial_{k'}$, $e=x^\gamma\partial_0$, where γ satisfies $\gamma_k-\gamma_{k'}=1$ such that $[h,e]=e$ in the generalized Cartan type* **S** *Lie algebra $x^\eta\mathbf{S}$ over $\mathbb{F}$, there exists a structure of noncommutative and noncocommutative Hopf algebra $(U(x^\eta\mathbf{S})[[t]],m,\iota,\Delta,S,\varepsilon)$ on $U(x^\eta\mathbf{S})[[t]]$ over $\mathbb{F}[[t]]$ with $U(x^\eta\mathbf{S})[[t]]/tU(x^\eta\mathbf{S})[[t]]\cong U(x^\eta\mathbf{S})$, which leaves the product of $U(x^\eta\mathbf{S})[[t]]$ undeformed but with the deformed coproduct, antipode and counit defined by*

$$\Delta(x^\alpha\partial)=x^\alpha\partial\otimes(1-et)^{\alpha_k-\alpha_{k'}}+\sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot d^{(\ell)}(x^\alpha\partial)t^\ell, \tag{22}$$

$$S(x^\alpha\partial)=-(1-et)^{-(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}(x^\alpha\partial)\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{23}$$

$$\varepsilon(x^\alpha\partial)=0, \tag{24}$$

where $x^\alpha\partial\in x^\alpha T_{\alpha-\eta}$.

Proof. By Lemmas 1.4 and 1.6, it follows from (19) and (21) that

$$\begin{aligned}
\Delta(x^\alpha\partial)&=\mathcal{F}\cdot\Delta_0(x^\alpha\partial)\cdot\mathcal{F}^{-1}\\
&=\mathcal{F}\cdot(x^\alpha\partial\otimes 1)\cdot F+\mathcal{F}\cdot(1\otimes x^\alpha\partial)\cdot F\\
&=\Big(\mathcal{F}F_{\alpha_{k'}-\alpha_k}\Big)\cdot(x^\alpha\partial\otimes 1)+\sum_{\ell=0}^{\infty}(-1)^\ell\Big(\mathcal{F}F_\ell\Big)\cdot\Big(h^{\langle\ell\rangle}\otimes d^{(\ell)}(x^\alpha\partial)t^\ell\Big)\\
&=\Big(1\otimes(1-et)^{\alpha_k-\alpha_{k'}}\Big)\cdot(x^\alpha\partial\otimes 1)\\
&\quad+\sum_{\ell=0}^{\infty}(-1)^\ell\Big(1\otimes(1-et)^{-\ell}\Big)\cdot\Big(h^{\langle\ell\rangle}\otimes d^{(\ell)}(x^\alpha\partial)t^\ell\Big)\\
&=x^\alpha\partial\otimes(1-et)^{\alpha_k-\alpha_{k'}}+\sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot d^{(\ell)}(x^\alpha\partial)t^\ell.
\end{aligned}$$

By (20) and Lemma 1.6, we obtain

$$\begin{aligned}
S(x^\alpha\partial)&=u^{-1}S_0(x^\alpha\partial)\,u=-v\cdot x^\alpha\partial\cdot u\\
&=-v\cdot u_{\alpha_k-\alpha_{k'}}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}(x^\alpha\partial)\cdot h_1^{\langle\ell\rangle}t^\ell\Big)\\
&=-(1-et)^{-(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}(x^\alpha\partial)\cdot h_1^{\langle\ell\rangle}t^\ell\Big).
\end{aligned}$$

Hence, we get the result. □

For later use, we need to make the following

Lemma 2.5. *For $s\geq 1$, one has*

(i) $$\Delta((x^\alpha\partial)^s)=\sum_{\substack{0\le j\le s\\ \ell\ge 0}}\binom{s}{j}(-1)^\ell(x^\alpha\partial)^j h^{\langle\ell\rangle}\otimes(1-et)^{j(\alpha_k-\alpha_{k'})-\ell}d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell.$$

(ii) $$S((x^\alpha\partial)^s)=(-1)^s(1-et)^{-s(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big).$$

Proof. By (19), (21) and Lemma 1.6, we obtain

$$\begin{aligned}\Delta((x^\alpha\partial)^s)&=\mathcal{F}\Big(x^\alpha\partial\otimes 1+1\otimes x^\alpha\partial\Big)^s\mathcal{F}^{-1}\\&=\sum_{j=0}^{s}\binom{s}{j}\mathcal{F}F_{j(\alpha_{k'}-\alpha_k)}(x^\alpha\partial\otimes 1)^j\Big(\sum_{\ell\geq 0}(-1)^\ell\mathcal{F}F_\ell\big(h^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell\big)\Big)\\&=\sum_{j=0}^{s}\sum_{\ell\geq 0}\binom{s}{j}(-1)^\ell\big((x^\alpha\partial)^j\otimes(1-et)^{j(\alpha_k-\alpha_{k'})-\ell}\big)\big(h^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell\big)\\&=\sum_{\substack{0\leq j\leq s\\ \ell\geq 0}}\binom{s}{j}(-1)^\ell(x^\alpha\partial)^jh^{\langle\ell\rangle}\otimes(1-et)^{j(\alpha_k-\alpha_{k'})-\ell}d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell.\end{aligned}$$

Again by (20) and Lemma 1.6, we get

$$\begin{aligned}S((x^\alpha\partial)^s)&=u^{-1}S_0((x^\alpha\partial)^s)\,u=(-1)^sv\cdot(x^\alpha\partial)^s\cdot u\\&=(-1)^sv\cdot u_{s(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big)\\&=(-1)^s(1-et)^{-s(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big).\end{aligned}$$

So this completes the proof. $\square$

2.3. Quantization integral forms of $\mathbb{Z}$-form $\mathbf{S}_{\mathbb{Z}}^+$ in char 0. As we known, $\{\alpha_nx^{\alpha-\epsilon_n}D_i-\alpha_ix^{\alpha-\epsilon_i}D_n=x^{\alpha-\epsilon_i-\epsilon_n}(\alpha_n\partial_i-\alpha_i\partial_n)\mid\alpha\in\mathbb{Z}_+^n,1\leq i<n\}$ is a $\mathbb{Z}$-basis of $\mathbf{S}_{\mathbb{Z}}^+$, as a subalgebra of both the simple Lie $\mathbb{Z}$-algebras $x^{-\underline{1}}\mathbf{S}_{\mathbb{Z}}$ and $\mathbf{W}_{\mathbb{Z}}^+$. In order to get the quantization integral forms of $\mathbb{Z}$-form $\mathbf{S}_{\mathbb{Z}}^+$, it suffices to consider what conditions are needed for those coefficients occurred in the formulae (22) & (23) to be integral for the indicated basis elements.

Lemma 2.6. ([**11**]) *For any a, k, $\ell\in\mathbb{Z}$, $a^\ell\prod_{j=0}^{\ell-1}(k+ja)/\ell!$ is an integer.* $\square$

From this Lemma (due to Grunspan), we see that if we take $\partial_0(\gamma)=\pm 1$, then A_ℓ and B_ℓ are integers in Theorem 2.4. In this paper, the cases we are interested in are: (i) $h=\partial_k-\partial_{k'}$, $e=x^{\epsilon_k}(\partial_k-2\partial_{k'})$ $(1\leq k\neq k'\leq n)$; (ii) $h=\partial_k-\partial_{k'}$, $e=x^{\epsilon_k-\epsilon_m}\partial_m$ $(1\leq k\neq k'\neq m\leq n)$. The latter will be discussed in Section 5. Denote by $\mathcal{F}(k,k')$ the corresponding Drinfel'd twist in the case (i). As a result of Theorem 2.4, we have

Corollary 2.7. Fix distinguished elements $h:=\partial_k-\partial_{k'}$, $e:=x^{\epsilon_k}(\partial_k-2\partial_{k'})$ $(1\leq k\neq k'\leq n)$, the corresponding quantization of $U(\mathbf{S}_{\mathbb{Z}}^+)$ over $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$ by Drinfel'd twist $\mathcal{F}(k,k')$ with the product undeformed is given by

$$\begin{aligned}\Delta(x^\alpha\partial)=x^\alpha\partial\otimes(1-et)^{\alpha_k-\alpha_{k'}}+\sum_{\ell=0}^{\infty}(-1)^\ell\,h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\,\cdot\\ \cdot\,x^{\alpha+\ell\epsilon_k}\big(A_\ell\partial-B_\ell(\partial_k-2\partial_{k'})\big)t^\ell,\end{aligned}\tag{25}$$

$$S(x^\alpha\partial)=-(1-et)^{-(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\epsilon_k}\big(A_\ell\partial-B_\ell(\partial_k-2\partial_{k'})\big)\cdot h_1^{\langle\ell\rangle}t^\ell\Big),\tag{26}$$

$$\varepsilon(x^\alpha\partial)=0, \tag{27}$$

where $A_\ell=\frac{1}{\ell!}\prod_{j=0}^{\ell-1}(\alpha_k-2\alpha_{k'}+j)$, $B_\ell=\partial(\epsilon_k)A_{\ell-1}$ with $A_0=1, A_{-1}=0$.

Remark 2.8. We get $n(n-1)$ *basic Drinfel'd twists* $\mathcal{F}(1,2),\cdots,\mathcal{F}(1,n)$, $\mathcal{F}(2,1)$, $\cdots,\mathcal{F}(n,n-1)$ over $U(\mathbf{S}_{\mathbb{Z}}^+)$. It is interesting to consider the products of some *basic Drinfel'd twists*, using the same argument as the proof of Theorem 2.4, one can get many more new Drinfel'd twists (which depends on a bit more calculations to be done), which will lead to many more new complicated quantizations not only over the $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$, but the possible quantizations over the $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ as well, via our modulo reduction approach developed in the next section.

3. Quantizations of the special algebra $\mathbf{S}(n;\underline{1})$

In this section, firstly, we make *modulo p reduction and base change with the $\mathcal{K}[[t]]$ replaced by $\mathcal{K}[t]$*, for the quantization of $U(\mathbf{S}_{\mathbb{Z}}^+)$ in char 0 (Corollary 2.7) to yield the quantization of $U(\mathbf{S}(n;\underline{1}))$, for the restricted simple modular Lie algebra $\mathbf{S}(n;\underline{1})$ in char p. Secondly, we shall further make *"p-restrictedness" reduction as well as base change with the $\mathcal{K}[t]$ replaced by $\mathcal{K}[t]_p^{(q)}$*, for the quantization of $U(\mathbf{S}(n;\underline{1}))$, which will lead to the required quantization of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$, the restricted universal enveloping algebra of $\mathbf{S}(n;\underline{1})$.

3.1. Modulo p reduction and base change. Let $\mathbb{Z}_p$ be the prime subfield of $\mathcal{K}$ with $\mathrm{char}(\mathcal{K})=p$. When considering $\mathbf{W}_{\mathbb{Z}}^+$ as a $\mathbb{Z}_p$-Lie algebra, namely, making modulo p reduction for the defining relations of $\mathbf{W}_{\mathbb{Z}}^+$, denoted by $\mathbf{W}_{\mathbb{Z}_p}^+$, we see that $(J_{\underline{1}})_{\mathbb{Z}_p}=\mathrm{Span}_{\mathbb{Z}_p}\{x^\alpha D_i\mid \exists\, j:\alpha_j\geq p\}$ is a maximal ideal of $\mathbf{W}_{\mathbb{Z}_p}^+$, and $\mathbf{W}_{\mathbb{Z}_p}^+/(J_{\underline{1}})_{\mathbb{Z}_p}\cong\mathbf{W}(n;\underline{1})_{\mathbb{Z}_p}=\mathrm{Span}_{\mathbb{Z}_p}\{x^{(\alpha)}D_i\mid 0\leq\alpha\leq\tau, 1\leq i\leq n\}$. For the subalgebra $\mathbf{S}_{\mathbb{Z}}^+$, we have $\mathbf{S}_{\mathbb{Z}_p}^+/(\mathbf{S}_{\mathbb{Z}_p}^+\cap(J_{\underline{1}})_{\mathbb{Z}_p})\cong\mathbf{S}'(n;\underline{1})_{\mathbb{Z}_p}$. We denote simply $\mathbf{S}_{\mathbb{Z}_p}^+\cap(J_{\underline{1}})_{\mathbb{Z}_p}$ as $(J_{\underline{1}}^+)_{\mathbb{Z}_p}$.

Moreover, we have $\mathbf{S}'(n;\underline{1})=\mathcal{K}\otimes_{\mathbb{Z}_p}\mathbf{S}'(n;\underline{1})_{\mathbb{Z}_p}=\mathcal{K}\mathbf{S}'(n;\underline{1})_{\mathbb{Z}_p}$, and $\mathbf{S}_{\mathcal{K}}^+=\mathcal{K}\mathbf{S}_{\mathbb{Z}_p}^+$.

Observe that the ideal $J_{\underline{1}}^+:=\mathcal{K}(J_{\underline{1}}^+)_{\mathbb{Z}_p}$ generates an ideal of $U(\mathbf{S}_{\mathcal{K}}^+)$ over $\mathcal{K}$, denoted by $J:=J_{\underline{1}}^+U(\mathbf{S}_{\mathcal{K}}^+)$, where $\mathbf{S}_{\mathcal{K}}^+/J_{\underline{1}}^+\cong\mathbf{S}'(n;\underline{1})$. Based on the formulae (25) & (26), J is a Hopf ideal of $U(\mathbf{S}_{\mathcal{K}}^+)$ satisfying $U(\mathbf{S}_{\mathcal{K}}^+)/J\cong U(\mathbf{S}'(n;\underline{1}))$. Note that elements $\sum a_{i,\alpha}\frac{1}{\alpha!}x^\alpha D_i$ in $\mathbf{S}_{\mathcal{K}}^+$ for $0\leq\alpha\leq\tau$ will be identified with $\sum a_{i,\alpha}x^{(\alpha)}D_i$ in $\mathbf{S}'(n;\underline{1})$ and those in $J_{\underline{1}}$ with 0. Hence, by Lemma 1.2 and Corollary 2.7, we get the quantization of $U(\mathbf{S}'(n;\underline{1}))$ over $U_t(\mathbf{S}'(n;\underline{1})):=U(\mathbf{S}'(n;\underline{1}))\otimes_{\mathcal{K}}\mathcal{K}[t]$ (not necessarily in $U(\mathbf{S}'(n;\underline{1}))[[t]]$, as seen in formulae (28) & (29)) as follows.

Theorem 3.1. *Fix two distinguished elements $h:=D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e:=2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$ $(1\leq k\neq k'\leq n)$, the corresponding quantization of $U(\mathbf{S}'(n;\underline{1}))$ over $U_t(\mathbf{S}'(n;\underline{1}))$ with the product undeformed is given by*

$$\begin{aligned}\Delta(D_{ij}(x^{(\alpha)}))&=D_{ij}(x^{(\alpha)})\otimes(1-et)^{\alpha_k-\delta_{ik}-\delta_{jk}-\alpha_{k'}+\delta_{ik'}+\delta_{jk'}}\\&\quad+\sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\Big(\bar{A}_\ell D_{ij}(x^{(\alpha+\ell\epsilon_k)})\\&\quad+\bar{B}_\ell(\delta_{ik}D_{k'j}+\delta_{jk}D_{ik'})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})})\Big)t^\ell,\end{aligned}\tag{28}$$

$$S(D_{ij}(x^{(\alpha)})) = -(1-et)^{-\alpha_k+\delta_{ik}+\delta_{jk}+\alpha_{k'}-\delta_{ik'}-\delta_{jk'}} \cdot \Big(\sum_{\ell=0}^{p-1}\big(\bar{A}_\ell D_{ij}(x^{(\alpha+\ell\epsilon_k)}) + \bar{B}_\ell(\delta_{ik}D_{k'j}+\delta_{jk}D_{ik'})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})})\big)\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{29}$$

$$\varepsilon(D_{ij}(x^{(\alpha)})) = 0, \tag{30}$$

$$\Delta(x^{(\tau-(p-1)\epsilon_j)}D_j) = x^{(\tau-(p-1)\epsilon_j)}D_j\otimes(1-et)^{p\,(\delta_{jk'}-\delta_{jk})} + 1\otimes x^{(\tau-(p-1)\epsilon_j)}D_j, \tag{31}$$

$$S(x^{(\tau-(p-1)\epsilon_j)}D_j) = -(1-et)^{p\,(\delta_{jk}-\delta_{jk'})}x^{(\tau-(p-1)\epsilon_j)}D_j, \tag{32}$$

$$\varepsilon(x^{(\tau-(p-1)\epsilon_j)}D_j) = 0, \tag{33}$$

where $0\le\alpha\le\tau$, $1\le j<i\le n$, $\bar{A}_\ell = \ell!\binom{\alpha_k+\ell}{\ell}(A_\ell-\delta_{jk}A_{\ell-1}-\delta_{ik}A_{\ell-1})\,(\mathrm{mod}\,p)$, $\bar{B}_\ell = 2\ell!\binom{\alpha_k+\ell-1}{\ell-1}(\alpha_{k'}+1)A_{\ell-1}\,(\mathrm{mod}\,p)$, $A_\ell = \frac{1}{\ell!}\prod_{m=0}^{\ell-1}(\alpha_k-\delta_{jk}-\delta_{ik}-2\alpha_{k'}+2\delta_{jk'}+2\delta_{ik'}+m)$ *and* $A_0=1, A_{-1}=0$.

Note that (28), (29) & (30) give the corresponding quantization of $U(\mathbf{S}(n;\underline{1}))$ over $U_t(\mathbf{S}(n;\underline{1})) := U(\mathbf{S}(n;\underline{1}))\otimes_{\mathcal{K}}\mathcal{K}[t]$ (also over $U(\mathbf{S}(n;\underline{1}))[[t]]$). It should be noticed that in this step — inducing from the quantization integral form of $U(\mathbf{S}_{\mathbb{Z}}^+)$ and making the modulo p reduction, we used the first base change with $\mathcal{K}[[t]]$ replaced by $\mathcal{K}[t]$, and the objects from $U(\mathbf{S}(n;\underline{1}))[[t]]$ turning to $U_t(\mathbf{S}(n;\underline{1}))$.

3.2. Modulo "p-restrictedness" reduction and base change. Let I be the ideal of $U(\mathbf{S}(n;\underline{1}))$ over $\mathcal{K}$ generated by $(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})$ and $(D_{ij}(x^{(\alpha)}))^p$ with $\alpha\ne\epsilon_i+\epsilon_j$ for $0\le\alpha\le\tau$ and $1\le j<i\le n$. $\mathbf{u}(\mathbf{S}(n;\underline{1})) = U(\mathbf{S}(n;\underline{1}))/I$ is of dimension $p^{(n-1)(p^n-1)}$. In order to get a reasonable quantization of finite dimension for $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ in char p, at first, it is necessary to clarify in concept the underlying vector space in which the required t-deformed object exists. According to our modular reduction approach, it should start to be induced from the $\mathcal{K}[t]$-algebra $U_t(\mathbf{S}(n;\underline{1}))$ in Theorem 3.1.

Firstly, we observe the following fact

LEMMA 3.2. (i) $(1-et)^p\equiv 1 \pmod{p, I}$.

(ii) $(1-et)^{-1}\equiv 1+et+\cdots+e^{p-1}t^{p-1} \pmod{p, I}$.

(iii) $h_a^{\langle\ell\rangle}\equiv 0 \pmod{p, I}$ *for* $\ell\ge p$, *and* $a\in\mathbb{Z}_p$.

PROOF. (i), (ii) follow from $e^p=0$ in $\mathbf{u}(\mathbf{S}(n;\underline{1}))$.

(iii) For $\ell\in\mathbb{Z}_+$, there is a unique decomposition $\ell=\ell_0+\ell_1 p$ with $0\le\ell_0<p$ and $\ell_1\ge 0$. Using the formulae (4) & (6), we have

$$\begin{aligned} h_a^{\langle\ell\rangle} &= h_a^{\langle\ell_0\rangle}\cdot h_{a+\ell_0}^{\langle\ell_1 p\rangle} \equiv h_a^{\langle\ell_0\rangle}\cdot(h_{a+\ell_0}^{\langle p\rangle})^{\ell_1} \qquad (\mathrm{mod}\ p)\\ &\equiv h_a^{\langle\ell_0\rangle}\cdot(h^p-h)^{\ell_1} \quad (\mathrm{mod}\ p), \end{aligned}$$

where we used the facts that $(x+1)(x+2)\cdots(x+p-1)\equiv x^{p-1}-1$ (mod p), and $(x+a+\ell_0)^p\equiv x^p+a+\ell_0$ (mod p). Hence, $h_a^{\langle\ell\rangle}\equiv 0$ (mod p, I) for $\ell\ge p$. □

The above Lemma, together with Theorem 3.1, indicates that the required t-deformation of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ (if it exists) in fact only happens in a p-truncated polynomial ring (with degrees of t less than p) with coefficients in $\mathbf{u}(\mathbf{S}(n;\underline{1}))$, i.e., $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})) := \mathbf{u}(\mathbf{S}(n;\underline{1}))\otimes_{\mathcal{K}}\mathcal{K}[t]_p^{(q)}$ (rather than in $\mathbf{u}_t(\mathbf{S}(n;\underline{1})) := \mathbf{u}(\mathbf{S}(n;\underline{1}))\otimes_{\mathcal{K}}$

$\mathcal{K}[t]$), where $\mathcal{K}[t]_p^{(q)}$ is taken to be a p-truncated polynomial ring which is a quotient of $\mathcal{K}[t]$ defined as

$$\mathcal{K}[t]_p^{(q)} = \mathcal{K}[t]/(t^p - qt), \qquad \textit{for} \ \ q \in \mathcal{K}. \tag{34}$$

Thereby, we obtain the underlying ring for our required t-deformation of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ over $\mathcal{K}[t]_p^{(q)}$, and $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})) = p\cdot\dim_{\mathcal{K}}\mathbf{u}(\mathbf{S}(n;\underline{1})) = p^{1+(n-1)(p^n-1)}$. Via modulo "restrictedness" reduction, it is necessary for us to work over the objects from $U_t(\mathbf{S}(n;\underline{1}))$ passage to $U_{t,q}(\mathbf{S}(n;\underline{1}))$ first, and then to $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ (see the proof of Theorem 3.5 below), here we used the second base change with $\mathcal{K}[t]_p^{(q)}$ instead of $\mathcal{K}[t]$.

We are now in a position to describe the following

Definition 3.3. With notations as above. A Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)$ over a ring $\mathcal{K}[t]_p^{(q)}$ of characteristic p is said to be a finite-dimensional quantization of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ if its Hopf algebra structure, via modular reduction and base changes, inherits from a twisting of the standard Hopf algebra $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$ by a Drinfeld twist such that $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))/t\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})) \cong \mathbf{u}(\mathbf{S}(n;\underline{1}))$.

To describe $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ explicitly, we still need an auxiliary Lemma.

Lemma 3.4. *Let* $e = 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$ *and* $d^{(\ell)} = \frac{1}{\ell!}(\mathrm{ad}\, e)^\ell$. *Then*

(i) $d^{(\ell)}(D_{ij}(x^{(\alpha)})) = \bar{A}_\ell D_{ij}(x^{(\alpha+\ell\epsilon_k)}) + \bar{B}_\ell(\delta_{ik}D_{k'j}+\delta_{jk}D_{ik'})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})})$, *where* $\bar{A}_\ell, \bar{B}_\ell$ *as in Theorem 3.1.*

(ii) $d^{(\ell)}(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = \delta_{\ell,0}D_{ij}(x^{(\epsilon_i+\epsilon_j)}) - \delta_{1,\ell}(\delta_{ik} - \delta_{jk})e$.

(iii) $d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) = \delta_{\ell,0}(D_{ij}(x^{(\alpha)}))^p - \delta_{1,\ell}(\delta_{ik} - \delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j}e$.

Proof. (i) Note that $A_\ell = \frac{1}{\ell!}\prod_{m=0}^{\ell-1}(\alpha_k - \delta_{jk} - \delta_{ik} - 2\alpha_{k'} + 2\delta_{jk'} + 2\delta_{ik'} + m)$, for $0 \le \alpha \le \tau$. By (17) and Theorem 3.1, we get

$$\begin{aligned}
d^{(\ell)}(D_{ij}(x^{(\alpha)})) &= \frac{1}{\alpha!}d^{(\ell)}(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i - \alpha_i\partial_j)) \\
&= \frac{1}{\alpha!}x^{\alpha-\epsilon_i-\epsilon_j+\ell\epsilon_k}(A_\ell(\alpha_j\partial_i - \alpha_i\partial_j) - (\alpha_j\delta_{ik} - \alpha_i\delta_{jk})A_{\ell-1}(\partial_k - 2\partial_{k'})) \\
&= \bar{A}_\ell D_{ij}(x^{(\alpha+\ell\epsilon_k)}) + \bar{B}_\ell(\delta_{ik}D_{k'j} + \delta_{jk}D_{ik'})(x^{(\alpha+(\ell-1)\epsilon_k+\epsilon_{k'})}).
\end{aligned}$$

(ii) Note that $A_0 = 1$ and $A_\ell = 0$ for $\ell \ge 1$,

$$\begin{aligned}
\bar{A}_\ell &= \ell!\binom{\alpha_k+\ell}{\ell}(A_\ell - \delta_{jk}A_{\ell-1} - \delta_{ik}A_{\ell-1}) \pmod p, \\
\bar{B}_\ell &= 2\ell!\binom{\alpha_k+\ell-1}{\ell-1}(\alpha_{k'}+1)A_{\ell-1} \pmod p.
\end{aligned}$$

We obtain $\bar{A}_0 = 1$ and $\bar{B}_0 = 0$. We also obtain $\bar{A}_1 = -(\delta_{ik} + \delta_{jk})(\alpha_k + 1)$, $\bar{B}_1 = 2(\alpha_{k'} + 1)$ and $\bar{A}_\ell = \bar{B}_\ell = 0$ for $\ell \ge 2$, namely, $d^{(\ell)}(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = 0$ for $\ell \ge 2$. So by (i), we have

$$\begin{aligned}
d^{(1)}(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &= -(\delta_{ik} + \delta_{jk})(\alpha_k + 1)D_{ij}(x^{(\epsilon_i+\epsilon_j+\epsilon_k)}) \\
&\quad + 2(\alpha_{k'} + 1)(\delta_{ik}D_{k'j} + \delta_{jk}D_{ik'})(x^{(\epsilon_i+\epsilon_j+\epsilon_{k'})}) \\
&= -(\delta_{ik} - \delta_{jk})e.
\end{aligned}$$

In any case, we arrive at the result as required.

(iii) From (15), we obtain that for $0 \leq \alpha \leq \tau$,

$$\begin{aligned} d^{(1)}\left((D_{ij}(x^{(\alpha)}))^p\right) &= \frac{1}{(\alpha!)^p}\big[\, e, (D_{ij}(x^{\alpha}))^p\,\big] = \frac{1}{(\alpha!)^p}\big[\, e, (x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i-\alpha_i\partial_j))^p\,\big] \\ &= \frac{1}{(\alpha!)^p}\sum_{\ell=1}^{p}(-1)^{\ell}\binom{p}{\ell}(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i-\alpha_i\partial_j))^{p-\ell} \\ &\qquad \cdot x^{\epsilon_k+\ell(\alpha-\epsilon_i-\epsilon_j)}(a_\ell(\partial_k-2\partial_{k'})-b_\ell(\alpha_j\partial_i-\alpha_i\partial_j)) \\ &\equiv -\frac{a_p}{\alpha!}\,x^{2\epsilon_k+p(\alpha-\epsilon_i-\epsilon_j)}(\partial_k-2\partial_{k'}) \qquad (\mathrm{mod}\ \ p\,) \\ &\equiv \begin{cases} -a_p\, e, & \textit{if} \quad \alpha=\epsilon_i+\epsilon_j \\ 0, & \textit{if} \quad \alpha\neq\epsilon_i+\epsilon_j \end{cases} \qquad (\mathrm{mod}\ \ J), \end{aligned}$$

where the last "$\equiv$" by using the identification w.r.t. modulo the ideal J as before, and $a_\ell = \prod_{m=0}^{\ell-1}(\alpha_j\partial_i-\alpha_i\partial_j)(\epsilon_k+m(\alpha-\epsilon_i-\epsilon_j))$, $b_\ell = \ell\,(\partial_k-2\partial_{k'})(\alpha-\epsilon_i-\epsilon_j)a_{\ell-1}$, and $a_p = \delta_{ik}-\delta_{jk}$ for $\alpha=\epsilon_i+\epsilon_j$.

Consequently, by the definition of $d^{(\ell)}$, we get $d^{(\ell)}((x^{(\alpha)}D_i)^p)=0$ in $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ for $2\leq\ell\leq p-1$ and $0\leq\alpha\leq\tau$. $\square$

Based on Theorem 3.1, Definition 3.3 and Lemma 3.4, we arrive at

THEOREM 3.5. *Fix two distinguished elements* $h := D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e := 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$ $(1\leq k\neq k'\leq n)$, *there is a noncommutative and noncocummtative Hopf algebra* $(\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)$ *over* $\mathcal{K}[t]_p^{(q)}$ *with its algebra structure undeformed, whose coalgebra structure is given by*

$$\begin{aligned} \Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)})&\otimes(1-et)^{\alpha_k-\delta_{ik}-\delta_{jk}-\alpha_{k'}+\delta_{ik'}+\delta_{jk'}} \\ &+\sum_{\ell=0}^{p-1}(-1)^{\ell}h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}d^{(\ell)}(D_{ij}(x^{(\alpha)}))t^{\ell}, \end{aligned} \tag{36}$$

(37)
$$S(D_{ij}(x^{(\alpha)})) = -(1-et)^{-\alpha_k+\delta_{ik}+\delta_{jk}+\alpha_{k'}-\delta_{ik'}-\delta_{jk'}}\cdot\Big(\sum_{\ell=0}^{p-1}d^{(\ell)}(D_{ij}(x^{(\alpha)}))\cdot h_1^{\langle\ell\rangle}t^{\ell}\Big),$$

$$\varepsilon(D_{ij}(x^{(\alpha)}))=0, \tag{38}$$

for $0\leq\alpha\leq\tau$, *which is finite dimensional with* $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})) = p^{1+(n-1)(p^n-1)}$.

PROOF. Set $U_{t,q}(\mathbf{S}(n;\underline{1})) := U(\mathbf{S}(n;\underline{1}))\otimes_{\mathcal{K}}\mathcal{K}[t]_p^{(q)}$. Note that the result of Theorem 3.1, via the base change with $\mathcal{K}[t]$ instead of $\mathcal{K}[t]_p^{(q)}$, is still valid over $U_{t,q}(\mathbf{S}(n;\underline{1}))$. Denote by $I_{t,q}$ the ideal of $U_{t,q}(\mathbf{S}(n;\underline{1}))$ over the ring $\mathcal{K}[t]_p^{(q)}$ generated by the same generators of the ideal I in $U(\mathbf{S}(n;\underline{1}))$ via the base change with $\mathcal{K}$ replaced by $\mathcal{K}[t]_p^{(q)}$. We shall show that $I_{t,q}$ is a Hopf ideal of $U_{t,q}(\mathbf{S}(n;\underline{1}))$. It suffices to verify that Δ and S preserve the generators in $I_{t,q}$ of $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

(I) By Lemmas 2.5, 3.2 & 3.4 (iii), we obtain

$$\Delta((D_{ij}(x^{(\alpha)}))^p) = (D_{ij}(x^{(\alpha)}))^p \otimes (1-et)^{p\,(\alpha_k-\alpha_{k'})} + \sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle} \otimes (1-et)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \tag{39}$$

$$\equiv (D_{ij}(x^{(\alpha)}))^p\otimes 1 + \sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \pmod p$$
$$= (D_{ij}(x^{(\alpha)}))^p\otimes 1 + 1\otimes(D_{ij}(x^{(\alpha)}))^p + h\otimes(1-et)^{-1}(\delta_{ik}-\delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j}et.$$

Hence, when $\alpha \neq \epsilon_i + \epsilon_j$, we get

$$\Delta((D_{ij}(x^{(\alpha)}))^p) = (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q};$$

and when $\alpha = \epsilon_i + \epsilon_j$, by Lemma 3.4 (ii), (28) becomes

$$\Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = D_{ij}(x^{(\epsilon_i+\epsilon_j)})\otimes 1 + 1\otimes D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + h\otimes(1-et)^{-1}(\delta_{ik}-\delta_{jk})et.$$

Combining with (39), we obtain

$$\begin{aligned}\Delta((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &\equiv ((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \otimes 1 \\ &\quad + 1 \otimes ((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q}.\end{aligned}$$

Thereby, we prove that the ideal $I_{t,q}$ is also a coideal of the Hopf algebra $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

(II) By Lemmas 2.5, 3.2 & 3.4 (iii), we have

$$S((D_{ij}(x^{(\alpha)}))^p) = -(1-et)^{-p(\alpha_k-\alpha_{k'})}\sum_{\ell=0}^{\infty} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h_1^{\langle\ell\rangle}t^\ell$$
$$\equiv -(D_{ij}(x^{(\alpha)}))^p - \sum_{\ell=1}^{p-1} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h_1^{\langle\ell\rangle}t^\ell \pmod p \tag{40}$$
$$= -(D_{ij}(x^{(\alpha)}))^p + (\delta_{ik}-\delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j}e\cdot h_1^{\langle 1\rangle}t.$$

Hence, when $\alpha \neq \epsilon_i + \epsilon_j$, we get

$$S\big((D_{ij}(x^{(\alpha)}))^p\big) = -(D_{ij}(x^{(\alpha)}))^p \in I_{t,q}.$$

When $\alpha = \epsilon_i + \epsilon_j$, by Lemma 3.4 (ii), (29) reads as

$$S(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = -D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + (\delta_{ik}-\delta_{jk})e\cdot h_1^{\langle 1\rangle}t.$$

Combining with (40), we obtain

$$S\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) = -\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) \in I_{t,q}.$$

Thereby, the ideal $I_{t,q}$ is indeed preserved by the antipode S of the quantization $U_{t,q}(\mathbf{S}(n;\underline{1}))$, the same as in Theorem 3.1.

(III) It is obvious to notice that $\varepsilon((D_{ij}(x^{(\alpha)}))^p) = 0$ for all α with $0 \le \alpha \le \tau$.

In other words, we prove that $I_{t,q}$ is a Hopf ideal in $U_{t,q}(\mathbf{S}(n;\underline{1}))$. We thus obtain the required t-deformation on $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$, for the Cartan type simple modular restricted Lie algebra of $\mathbf{S}$ type — the special algebra $\mathbf{S}(n;\underline{1})$. $\square$

Remark 3.6. (i) Set $f=(1-et)^{-1}$. By Lemma 3.4 & Theorem 3.5, one gets

$$[h,f]=f^2-f,\quad h^p=h,\quad f^p=1,\quad \Delta(h)=h\otimes f+1\otimes h,$$

where f is a group-like element, and $S(h)=-hf^{-1}$, $\varepsilon(h)=0$. So the subalgebra generated by h and f is a Hopf subalgebra of $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$, which is isomorphic to the well-known Radford Hopf algebra over $\mathcal{K}$ in char p (see [**21**]).

(ii) According to our argument, given a parameter $q\in\mathcal{K}$, one can specialize t to any root of the p-polynomial $t^p-qt\in\mathcal{K}[t]$ in a split field of $\mathcal{K}$. For instance, if take $q=1$, then one can specialize t to any scalar in $\mathbb{Z}_p$. If set $t=0$, then we get the original standard Hopf algebra structure of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$. In this way, we indeed get a new Hopf algebra structure over the same restricted universal enveloping algebra $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ over $\mathcal{K}$ under the assumption that $\mathcal{K}$ is algebraically closed, which has the new coalgebra structure induced by Theorem 3.5, but has dimension $p^{(n-1)(p^n-1)}$.

4. More quantizations

In this section, we can get more Drinfel'd twists by considering the products of some pairwise different *basic Drinfel'd twists* as stated in Remark 2.8. By the same argument as in Theorem 2.4, one can get many more new complicated quantizations not only over the $U(\mathbf{S}^+_{\mathbb{Z}})[[t]]$, but over the $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ as well. Moreover, we prove that the twisted structures given by some products of pairwise different *basic Drinfel'd twists* with different length are nonisomorphic.

4.1. More Drinfel'd twists. We consider the products of pairwise different and mutually commutative basic Drinfel'd twists. Note that $[\mathcal{F}(i,j),\mathcal{F}(k,m)]=0$ for $i\neq k,m$ and $j\neq k$. This fact, according to the definition of $\mathcal{F}(k,m)$, implies the commutative relations in the case $i\neq k,m$ and $j\neq k$:

$$\begin{aligned}&(\mathcal{F}(k,m)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i,j))=(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i,j))(\mathcal{F}(k,m)\otimes 1),\\&(1\otimes\mathcal{F}(k,m))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i,j))=(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i,j))(1\otimes\mathcal{F}(k,m)),\end{aligned}\tag{41}$$

which give rise to the following property.

Theorem 4.1. *$\mathcal{F}(i,j)\mathcal{F}(k,m)(i\neq k,m;j\neq k)$ is still a Drinfel'd twist on $U(\mathbf{S}^+_{\mathbb{Z}})[[t]]$.*

Proof. Note that $\Delta_0\otimes\mathrm{id}$, $\mathrm{id}\otimes\Delta_0$, $\varepsilon_0\otimes\mathrm{id}$ and $\mathrm{id}\otimes\varepsilon_0$ are algebraic homomorphisms. According to Lemma 1.4, it suffices to check that

$$\begin{aligned}&(\mathcal{F}(i,j)\mathcal{F}(k,m)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i,j)\mathcal{F}(k,m))\\&\qquad=(1\otimes\mathcal{F}(i,j)\mathcal{F}(k,m))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i,j)\mathcal{F}(k,m)).\end{aligned}$$

Using (41), we have

$$\begin{aligned}\mathrm{LHS}&=(\mathcal{F}(i,j)\otimes 1)(\mathcal{F}(k,m)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i,j))(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(k,m))\\&=(\mathcal{F}(i,j)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i,j))(\mathcal{F}(k,m)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(k,m))\\&=(1\otimes\mathcal{F}(i,j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i,j))(1\otimes\mathcal{F}(k,m))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(k,m))\\&=(1\otimes\mathcal{F}(i,j))(1\otimes\mathcal{F}(k,m))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i,j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(k,m))=\mathrm{RHS}.\end{aligned}$$

This completes the proof. □

More generally, we have the following

Corollary 4.2. Let $\mathcal{F}(i_1,j_1),\cdots,\mathcal{F}(i_m,j_m)$ be m pairwise different basic Drinfel'd twists and $[\mathcal{F}(i_k,j_k),\mathcal{F}(i_s,j_s)]=0$ for all $1\le k\ne s\le m$. Then $\mathcal{F}(i_1,j_1)\cdots\mathcal{F}(i_m,j_m)$ is still a Drinfel'd twist.

We denote $\mathcal{F}_m=\mathcal{F}(i_1,j_1)\cdots\mathcal{F}(i_m,j_m)$ and the length of $\mathcal{F}(i_1,j_1)\cdots\mathcal{F}(i_m,j_m)$ as m. These twists lead to more quantizations.

4.2. More quantizations. We consider the modular reduction process for the quantizations of $U(\mathbf{S}^+)[[t]]$ arising from those products of some pairwise different and mutually commutative basic Drinfel'd twists. We will then get lots of new families of noncommutative and noncocommutative Hopf algebras of dimension $p^{1+(n-1)(p^n-1)}$ with indeterminate t or of dimension $p^{(n-1)(p^n-1)}$ with specializing t into a scalar in $\mathcal{K}$.

Let $A(k,k')_\ell,\ B(k,k')_\ell$ and $A(m,m')_n,\ B(m,m')_n$ denote the coefficients of the corresponding quantizations of $U(\mathbf{S}_{\mathbb{Z}}^+)$ over $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$ given by Drinfel'd twists $\mathcal{F}(k,k')$ and $\mathcal{F}(m,m')$ respectively as in Corollary 2.7. Note that $A(k,k')_0=A(m,m')_0=1$, $A(k,k')_{-1}=A(m,m')_{-1}=0$.

Set

$$\begin{aligned}\partial(m,m';k,k')_{\ell,n}:=A(m,m')_nA(k,k')_\ell\partial-A(m,m')_nB(k,k')_\ell(\partial_k-2\partial_{k'})\\ -A(k,k')_\ell B(m,m')_n(\partial_m-2\partial_{m'}).\end{aligned}$$

Lemma 4.3. *Fix distinguished elements $h(k,k')=\partial_k-\partial_{k'}$, $e(k,k')=x^{\epsilon_k}(\partial_k-2\partial_{k'})$ $(1\le k\ne k'\le n)$ and $h(m,m')=\partial_m-\partial_{m'}$, $e(m,m')=x^{\epsilon_m}(\partial_m-2\partial_{m'})$ $(1\le m\ne m'\le n)$ with $k\ne m,m'$ and $k'\ne m$, the corresponding quantization of $U(\mathbf{S}_{\mathbb{Z}}^+)$ over $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$ by Drinfel'd twist $\mathcal{F}=\mathcal{F}(m,m')\mathcal{F}(k,k')$ with the product undeformed is given by*

$$\begin{aligned}\Delta(x^\alpha\partial)&=x^\alpha\partial\otimes\big(1-e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\big(1-e(m,m')t\big)^{\alpha_m-\alpha_{m'}}\\&\quad+\sum_{n,\ell=0}^{\infty}(-1)^{n+\ell}h(k,k')^{\langle\ell\rangle}\cdot h(m,m')^{\langle n\rangle}\otimes\big(1-e(k,k')t\big)^{-\ell}\\&\quad\cdot\big(1-e(m,m')t\big)^{-n}x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}t^{n+\ell},\end{aligned}\tag{42}$$

$$\begin{aligned}S(x^\alpha\partial)&=-\big(1-e(k,k')t\big)^{-\alpha_k+\alpha_{k'}}\big(1-e(m,m')t\big)^{-\alpha_m+\alpha_{m'}}\cdot\\&\quad\cdot\sum_{n,\ell=0}^{\infty}x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}\cdot h(m,m')_1^{\langle n\rangle}h(k,k')_1^{\langle\ell\rangle}t^{n+\ell},\end{aligned}\tag{43}$$

$$\varepsilon(x^\alpha\partial)=0,\tag{44}$$

for $x^\alpha\partial\in\mathbf{S}_{\mathbb{Z}}^+$.

Proof. Using Corollary 2.7, we get

$$\begin{aligned}\Delta(x^\alpha\partial)&=\mathcal{F}(m,m')\mathcal{F}(k,k')\Delta_0(x^\alpha\partial)\mathcal{F}(k,k')^{-1}\mathcal{F}(m,m')^{-1}\\&=\mathcal{F}(m,m')\Big(x^\alpha\partial\otimes\big(1-e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\\&\quad+\sum_{\ell=0}^{\infty}(-1)^\ell h(k,k')^{\langle\ell\rangle}\otimes\big(1-e(k,k')t\big)^{-\ell}\cdot x^{\alpha+\ell\epsilon_k}\big(A(k,k')_\ell\partial\\&\quad-B(k,k')_\ell(\partial_k-2\partial_{k'})\big)t^\ell\Big)\mathcal{F}(m,m')^{-1}.\end{aligned}$$

Using (19) and Lemma 2.1, we get

$$\begin{aligned}
&\mathcal{F}(m,m')\Big(x^\alpha\partial\otimes\big(1{-}e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\Big)\mathcal{F}(m,m')^{-1}\\
&\qquad=\mathcal{F}(m,m')\Big(x^\alpha\partial\otimes 1\Big)\mathcal{F}(m,m')^{-1}\Big(1\otimes\big(1{-}e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\Big)\\
&\qquad=\mathcal{F}(m,m')\mathcal{F}(m,m')^{-1}_{\alpha_{m'}-\alpha_m}\Big(x^\alpha\partial\otimes 1\Big)\Big(1\otimes\big(1{-}e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\Big)\\
&\qquad=\Big(1\otimes\big(1{-}e(m,m')t\big)^{\alpha_m-\alpha_{m'}}\Big)\Big(x^\alpha\partial\otimes 1\Big)\Big(1\otimes\big(1{-}e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\Big)\\
&\qquad=x^\alpha\partial\otimes\big(1{-}e(k,k')t\big)^{\alpha_k-\alpha_{k'}}\big(1{-}e(m,m')t\big)^{\alpha_m-\alpha_{m'}}.
\end{aligned}$$

Using (21), we have

$$\begin{aligned}
&\mathcal{F}(m,m')\Big(\sum_{\ell=0}^{\infty}(-1)^\ell h(k,k')^{\langle\ell\rangle}\otimes\big(1{-}e(k,k')t\big)^{-\ell}\\
&\qquad\cdot x^{\alpha+\ell\epsilon_k}\big(A(k,k')_\ell\partial-B(k,k')_\ell(\partial_k{-}2\partial_{k'})\big)t^\ell\Big)\mathcal{F}(m,m')^{-1}\\
&\quad=\sum_{\ell=0}^{\infty}(-1)^\ell h(k,k')^{\langle\ell\rangle}\otimes\big(1{-}e(k,k')t\big)^{-\ell}\\
&\qquad\cdot\mathcal{F}(m,m')\Big(1\otimes x^{\alpha+\ell\epsilon_k}\big(A(k,k')_\ell\partial-B(k,k')_\ell(\partial_k{-}2\partial_{k'})\big)\Big)t^\ell\mathcal{F}(m,m')^{-1}\\
&\quad=\sum_{n,\ell=0}^{\infty}(-1)^\ell h(k,k')^{\langle\ell\rangle}\otimes\big(1{-}e(k,k')t\big)^{-\ell}\cdot\mathcal{F}(m,m')F(m,m')_n\Big(h(m,m')^{\langle n\rangle}\otimes\\
&\qquad\quad x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}t^{n+\ell}\Big)\\
&\quad=\sum_{n,\ell=0}^{\infty}(-1)^\ell h(k,k')^{\langle\ell\rangle}h(m,m')^{\langle n\rangle}\otimes\big(1{-}e(k,k')t\big)^{-\ell}\big(1{-}e(m,m')t\big)^{-n}\\
&\qquad\cdot x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}t^{n+\ell}.
\end{aligned}$$

For $k\neq m,m'$ and $k'\neq m$, by the definitions of v and u, we get

$$\begin{aligned}
v&=v(k,k')v(m,m')=v(m,m')v(k,k'),\\
u&=u(m,m')u(k,k')=u(k,k')u(m,m').
\end{aligned}$$

Note $u(m,m')h(k,k')=h(k,k')u(m,m')$, $v(m,m')e(k,k')=e(k,k')v(m,m')$. By Corollary 2.7 and (20), we have

$$\begin{aligned}
S(x^\alpha\partial)&=-v\cdot x^\alpha\partial\cdot u\\
&=-v(m,m')v(k,k')\cdot x^\alpha\partial\cdot u(k,k')u(m,m')\\
&=v(m,m')\cdot\Big(-\big(1{-}e(k,k')t\big)^{-\alpha_k+\alpha_{k'}}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\epsilon_k}\big(A(k,k')_\ell\partial\\
&\qquad-B(k,k')_\ell(\partial_k{-}2\partial_{k'})\big)\cdot h(k,k')_1^{\langle\ell\rangle}t^\ell\Big)\Big)\cdot u(m,m')\\
&=-\big(1{-}e(k,k')t\big)^{-\alpha_k+\alpha_{k'}}\cdot v(m,m')u(m,m')_{\alpha_m-\alpha_{m'}}\\
&\qquad\cdot\sum_{n,\ell=0}^{\infty}x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}\cdot h(m,m')_1^{\langle n\rangle}h(k,k')_1^{\langle\ell\rangle}t^{n+\ell}
\end{aligned}$$

$$= -\big(1-e(k,k')t\big)^{-\alpha_k+\alpha_{k'}}\big(1-e(m,m')t\big)^{-\alpha_m+\alpha_{m'}}$$
$$\cdot\sum_{n,\ell=0}^{\infty} x^{\alpha+\ell\epsilon_k+n\epsilon_m}\partial(m,m';k,k')_{\ell,n}\cdot h(m,m')_1^{\langle n\rangle}h(k,k')_1^{\langle \ell\rangle}t^{n+\ell}.$$

This completes the proof. □

Set $\alpha(k,k') = \alpha_k-\delta_{ik}-\delta_{jk}-\alpha_{k'}+\delta_{ik'}+\delta_{jk'}$ and $d^{(\ell)}_{kk'} = \frac{1}{\ell!}(\mathrm{ad}\, e(k,k'))^\ell$. Write coefficients $\bar{A}_\ell$, $\bar{B}_\ell$, A_ℓ in Theorem 3.1 as $\bar{A}(k,k')_\ell$, $\bar{B}(k,k')_\ell$, $A(k,k')_\ell$, respectively. Set

$$\begin{aligned}D_{ij}(m,m';k,k')_{\ell,n} := {}&\bar{A}(k,k')_\ell\bar{A}(m,m')_n D_{ij}(x^{(\alpha+\ell\epsilon_k+n\epsilon_m)})\\&+\bar{B}(k,k')_\ell\bar{A}(m,m')_n(\delta_{ik}D_{k'j}+\delta_{jk}D_{ik'})(x^{(\alpha+(\ell-1)\epsilon_k+n\epsilon_m+\epsilon_{k'})})\\&+\bar{A}(k,k')_\ell\bar{B}(m,m')_n(\delta_{im}D_{k'j}+\delta_{jm}D_{ik'})(x^{(\alpha+\ell\epsilon_k+(n-1)\epsilon_m+\epsilon_{k'})}).\end{aligned}$$

Using Lemma 4.3, we get a new quantization of $U(\mathbf{S}(n;\underline{1}))$ over $U_t(\mathbf{S}(n;\underline{1}))$ by Drinfel'd twist $\mathcal{F} = \mathcal{F}(m,m')\mathcal{F}(k,k')$ as follows.

Lemma 4.4. *Fix distinguished elements $h(k,k') = D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e(k,k') = 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$; $h(m,m') = D_{mm'}(x^{(\epsilon_m+\epsilon_{m'})})$, $e(m,m') = 2D_{mm'}(x^{(2\epsilon_m+\epsilon_{m'})})$ with $k\neq m,m'$; $k'\neq m$, the corresponding quantization of $U(\mathbf{S}(n;\underline{1}))$ on $U_t(\mathbf{S}(n;\underline{1}))$ (also on $U(\mathbf{S}(n;\underline{1}))[[t]]$) with the product undeformed is given by*

$$\begin{aligned}\Delta(D_{ij}(x^{(\alpha)})) = {}&D_{ij}(x^{(\alpha)})\otimes\big(1-e(k,k')t\big)^{\alpha(k,k')}\big(1-e(m,m')t\big)^{\alpha(m,m')}\\&+\sum_{n,\ell=0}^{p-1}(-1)^{n+\ell}h(k,k')^{\langle\ell\rangle}h(m,m')^{\langle n\rangle}\otimes\big(1-e(k,k')t\big)^{-\ell}\cdot\\&\cdot\big(1-e(m,m')t\big)^{-n}D_{ij}(m,m';k,k')_{\ell,n}t^{n+\ell},\end{aligned}\tag{45}$$

$$\begin{aligned}S(D_{ij}(x^{(\alpha)})) = {}&-\big(1-e(k,k')t\big)^{-\alpha(k,k')}\big(1-e(m,m')t\big)^{-\alpha(m,m')}\cdot\\&\cdot\Big(\sum_{n,\ell=0}^{p-1}D_{ij}(m,m';k,k')_{\ell,n}h(k,k')_1^{\langle\ell\rangle}h(m,m')_1^{\langle n\rangle}t^{n+\ell}\Big),\end{aligned}\tag{46}$$

$$\varepsilon(D_{ij}(x^{(\alpha)})) = 0,\tag{47}$$

where $0\leq\alpha\leq\tau$.

For the further discussion, we need two lemmas below about the quantization of $U(\mathbf{S}(n;\underline{1}))$ over $U(\mathbf{S}(n;\underline{1}))[[t]]$ in Lemma 4.4.

Lemma 4.5. *For $s\geq 1$, one has*

(i)
$$\begin{aligned}\Delta((D_{ij}(x^{(\alpha)}))^s) = {}&\sum_{\substack{0\leq j\leq s\\ n,\ell\geq 0}}\binom{s}{j}(-1)^{n+\ell}(D_{ij}(x^{(\alpha)}))^j h(k,k')^{\langle\ell\rangle}h(m,m')^{\langle n\rangle}\otimes\\&\big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell}\big(1-e(m,m')t\big)^{j\alpha(m,m')-n}\cdot\\&\cdot d^{(n)}_{mm'}d^{(\ell)}_{kk'}((D_{ij}(x^{(\alpha)}))^{s-j})t^{\ell+n}.\end{aligned}$$

(ii)
$$\begin{aligned}S((D_{ij}(x^{(\alpha)}))^s) = {}&(-1)^s\big(1-e(m,m')t\big)^{-s\alpha(m,m')}\big(1-e(k,k')t\big)^{-s\alpha(k,k')}\cdot\\&\cdot\Big(\sum_{n,\ell=0}^{\infty}d^{(n)}_{mm'}d^{(\ell)}_{kk'}((D_{ij}(x^{(\alpha)}))^s)h(k,k')_1^{\langle\ell\rangle}h(m,m')_1^{\langle n\rangle}t^{n+\ell}\Big).\end{aligned}$$

PROOF. By Lemma 2.5, (21), (19) and Lemma 1.6, we obtain

$$
\begin{aligned}
&\Delta((D_{ij}(x^{(\alpha)}))^s) = \mathcal{F}\Big(D_{ij}(x^{(\alpha)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\alpha)})\Big)^s \mathcal{F}^{-1} \\
&= \mathcal{F}(m,m')\Big(\sum_{\substack{0\le j\le s\\ \ell\ge 0}} \binom{s}{j}(-1)^\ell (D_{ij}(x^{(\alpha)}))^j h(k,k')^{\langle \ell\rangle} \otimes \big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell} \\
&\quad \cdot d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^{s-j})t^\ell\Big)\mathcal{F}(m,m')^{-1} \\
&= \mathcal{F}(m,m')\Big(\sum_{\substack{0\le j\le s\\ \ell\ge 0}} \binom{s}{j}(-1)^\ell \big((D_{ij}(x^{(\alpha)}))^j \otimes 1\big)\big(h(k,k')^{\langle \ell\rangle} \otimes \big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell}\big) \\
&\quad \cdot \big(1 \otimes d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^{s-j})t^\ell\big)\Big)\mathcal{F}(m,m')^{-1} \\
&= \mathcal{F}(m,m') \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}\big((D_{ij}(x^{(\alpha)}))^j \otimes 1\big)\mathcal{F}(m,m')^{-1}_n h(k,k')^{\langle \ell\rangle} h(m,m')^{\langle n\rangle} \\
&\quad \otimes \big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^{s-j})t^{\ell+n} \\
&= \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}\mathcal{F}(m,m')\mathcal{F}(m,m')^{-1}_{n-j\alpha(m,m')}\big((D_{ij}(x^{(\alpha)}))^j \otimes 1\big)h(k,k')^{\langle \ell\rangle} \\
&\quad \cdot h(m,m')^{\langle n\rangle} \otimes \big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^{s-j})t^{\ell+n} \\
&= \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}(D_{ij}(x^{(\alpha)}))^j h(k,k')^{\langle \ell\rangle} h(m,m')^{\langle n\rangle} \otimes \big(1-e(k,k')t\big)^{j\alpha(k,k')-\ell} \\
&\quad \cdot \big(1-e(m,m')t\big)^{j\alpha(m,m')-n} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^{s-j})t^{\ell+n}.
\end{aligned}
$$

Again by (20) and Lemma 1.6,

$$
\begin{aligned}
S((D_{ij}(x^{(\alpha)}))^s) &= u^{-1} S_0((D_{ij}(x^{(\alpha)}))^s)\, u \\
&= (-1)^s v \cdot (D_{ij}(x^{(\alpha)}))^s \cdot u \\
&= (-1)^s v(m,m')\Big(\big(1-e(k,k')t\big)^{-s\alpha(k,k')} \\
&\qquad \cdot \Big(\sum_{\ell=0}^{\infty} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^s) \cdot h(k,k')_1^{\langle \ell\rangle} t^\ell\Big)\Big)u(m,m') \\
&= (-1)^s v(m,m')u(m,m')_{s\alpha(m,m')}\big(1-e(k,k')t\big)^{-s\alpha(k,k')} \\
&\qquad \cdot \Big(\sum_{n,\ell=0}^{\infty} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^s) \cdot h(k,k')_1^{\langle \ell\rangle} h(m,m')_1^{\langle n\rangle} t^{n+\ell}\Big) \\
&= (-1)^s \big(1-e(m,m')t\big)^{-s\alpha(m,m')}\big(1-e(k,k')t\big)^{-s\alpha(k,k')} \\
&\qquad \cdot \Big(\sum_{n,\ell=0}^{\infty} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^s) \cdot h(k,k')_1^{\langle \ell\rangle} h(m,m')_1^{\langle n\rangle} t^{n+\ell}\Big).
\end{aligned}
$$

This completes the proof. □

LEMMA 4.6. *Set* $e(k,k') = 2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$, $e(m,m') = 2D_{mm'}(x^{(2\epsilon_m+\epsilon_{m'})})$,

$d_{kk'}^{(\ell)}=\frac{1}{\ell!}(\operatorname{ad}e(k,k'))^\ell$ *and* $d_{mm'}^{(n)}=\frac{1}{n!}(\operatorname{ad}e(m,m'))^n$. *Then*

(i) $d_{mm'}^{(n)}d_{kk'}^{(\ell)}(D_{ij}(x^{(\alpha)}))=D_{ij}(m,m';k,k')_{\ell,n}$,
where $D_{ij}(m,m';k,k')_{\ell,n}$ *as in Lemma 4.4.*

(ii) $$d_{mm'}^{(n)}d_{kk'}^{(\ell)}(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))=\delta_{\ell,0}\delta_{n,0}D_{ij}(x^{(\epsilon_i+\epsilon_j)})-\delta_{n,0}\delta_{1,\ell}(\delta_{ik}-\delta_{jk})e(k,k') - \delta_{\ell,0}\delta_{1,n}(\delta_{im}-\delta_{jm})e(m,m').$$

(iii) $$d_{mm'}^{(n)}d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)=\delta_{\ell,0}\delta_{n,0}(D_{ij}(x^{(\alpha)}))^p-\delta_{n,0}\delta_{1,\ell}(\delta_{ik}-\delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j}\cdot \cdot e(k,k')-\delta_{\ell,0}\delta_{1,n}(\delta_{im}-\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j}e(m,m').$$

Proof. (i) For $0\le\alpha\le\tau$, using (17), we obtain

$$\begin{aligned}
&d_{mm'}^{(n)}d_{kk'}^{(\ell)}(D_{ij}(x^{(\alpha)}))\\
&\quad=d_{mm'}^{(n)}d_{kk'}^{(\ell)}\Big(\frac{1}{\alpha!}x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i-\alpha_i\partial_j)\Big)\\
&\quad=d_{mm'}^{(n)}\Big(\frac{1}{\alpha!}x^{\alpha-\epsilon_i-\epsilon_j+\ell\epsilon_k}\big(A(k,k')_\ell(\alpha_j\partial_i-\alpha_i\partial_j)-B(k,k')_\ell(\partial_k-2\partial_{k'})\big)\Big)\\
&\quad=\frac{1}{\alpha!}x^{\alpha-\epsilon_i-\epsilon_j+\ell\epsilon_k+n\epsilon_m}\big(A(k,k')_\ell A(m,m')_n(\alpha_j\partial_i-\alpha_i\partial_j)\\
&\qquad-A(m,m')_nB(k,k')_\ell(\partial_k-2\partial_{k'})-A(k,k')_\ell B(m,m')_n(\partial_{k'}-2\partial_{k'})\big)\\
&\quad=D_{ij}(m,m';k,k')_{\ell,n}.
\end{aligned}$$

(ii), (iii) may be proved directly using Lemma 3.4. □

Using Lemmas 3.2, 3.4, 4.5 & 4.6, we get a new Hopf algebra structure over the same restricted universal enveloping algebra $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ over $\mathcal{K}$ by the products of two different and commutative basic Drinfel'd twists.

Theorem 4.7. *Fix two distinguished elements* $h(k,k'):=D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e(k,k'):=2D_{kk'}(x^{(2\epsilon_k+\epsilon_{k'})})$ $(1\le k\ne k'\le n)$ *and* $h(m,m'):=D_{mm'}(x^{(\epsilon_m+\epsilon_{m'})})$, $e(m,m'):=2D_{mm'}(x^{(2\epsilon_m+\epsilon_{m'})})$ $(1\le m\ne m'\le n)$ *with* $k\ne m,m'$; $k'\ne m$, *there is a noncommutative and noncocummtative Hopf algebra* $(\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})),m,\iota,\Delta,S,\varepsilon)$ *over* $\mathcal{K}[t]_p^{(q)}$ *with the product undeformed, whose coalgebra structure is given by*

$$\begin{aligned}
\Delta(D_{ij}(x^{(\alpha)}))&=D_{ij}(x^{(\alpha)})\otimes\big(1-e(k,k')t\big)^{\alpha(k,k')}\big(1-e(m,m')t\big)^{\alpha(m,m')}\\
&\quad+\sum_{n,\ell=0}^{p-1}(-1)^{\ell+n}h(k,k')^{\langle\ell\rangle}h(m,m')^{\langle n\rangle}\otimes\big(1-e(k,k')t\big)^{-\ell}\cdot\\
&\qquad\cdot\big(1-e(m,m')t\big)^{-n}d_{kk'}^{(\ell)}d_{mm'}^{(n)}(D_{ij}(x^{(\alpha)}))t^{\ell+n},
\end{aligned}\tag{48}$$

$$\begin{aligned}
S(D_{ij}(x^{(\alpha)}))&=-\big(1-e(k,k')t\big)^{-\alpha(k,k')}\big(1-e(m,m')t\big)^{-\alpha(m,m')}\cdot\\
&\quad\cdot\Big(\sum_{n,\ell=0}^{p-1}d_{kk'}^{(\ell)}d_{mm'}^{(n)}(D_{ij}(x^{(\alpha)}))h(k,k')_1^{\langle\ell\rangle}h(m,m')_1^{\langle n\rangle}t^{\ell+n}\Big),
\end{aligned}\tag{49}$$

$$\varepsilon(D_{ij}(x^{(\alpha)}))=0,\tag{50}$$

where $0\le\alpha\le\tau$, *and* $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))=p^{1+(n-1)(p^n-1)}$.

Proof. Let $I_{t,q}$ denote the ideal of $(U_{t,q}(\mathbf{S}(n;\underline{1})),m,\iota,\Delta,S,\varepsilon)$ over the ring $\mathcal{K}[t]_p^{(q)}$ generated by the same generators as in I $(q\in\mathcal{K})$. Observe that the result

in Lemma 4.4, via the base change with $\mathcal{K}[t]$ replaced by $\mathcal{K}[t]_p^{(q)}$, is still valid for $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

In what follows, we shall show that $I_{t,q}$ is a Hopf ideal of $U_{t,q}(\mathbf{S}(n;\underline{1}))$. To this end, it suffices to verify that Δ and S preserve the generators of $I_{t,q}$.

(I) By Lemmas 4.5, 3.2, 3.4 & 4.6, we obtain

$$\begin{aligned}
\Delta((D_{ij}(x^{(\alpha)}))^p) &= (D_{ij}(x^{(\alpha)}))^p \otimes (1-e(k,k')t)^{p\,\alpha(k,k')}(1-e(m,m')t)^{p\,\alpha(m,m')} \\
&\quad + \sum_{n,\ell=0}^{\infty} (-1)^{n+\ell} h(k,k')^{\langle \ell\rangle} h(m,m')^{\langle n\rangle} \otimes (1-e(k,k')t)^{-\ell} \\
&\quad \cdot (1-e(m,m')t)^{-n} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) t^{n+\ell} \\
&\equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + \sum_{n,\ell=0}^{p-1} (-1)^{n+\ell} h(k,k')^{\langle \ell\rangle} h(m,m')^{\langle n\rangle} \otimes (1-e(k,k')t)^{-\ell} \\
&\quad \cdot (1-e(m,m')t)^{-n} d_{mm'}^{(n)} d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) t^{n+\ell} \pmod p \\
&= (D_{ij}(x^{(\alpha)}))^p \otimes 1 + \sum_{n,\ell=0}^{p-1} (-1)^{n+\ell} h(k,k')^{\langle \ell\rangle} h(m,m')^{\langle n\rangle} \otimes (1-e(k,k')t)^{-\ell} \\
&\quad \cdot (1-e(m,m')t)^{-n} \Big(\delta_{\ell,0}\delta_{n,0}(D_{ij}(x^{(\alpha)}))^p - \delta_{n,0}\delta_{1,\ell}(\delta_{ik}-\delta_{jk}) \\
&\quad \cdot \delta_{\alpha,\epsilon_i+\epsilon_j} e(k,k') - \delta_{\ell,0}\delta_{1,n}(\delta_{im}-\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j} e(m,m')\Big) t^{n+\ell} \\
&= (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \\
&\quad + h(k,k') \otimes (1-e(k,k')t)^{-1}(\delta_{ik}-\delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j} e(k,k')t \\
&\quad + h(m,m') \otimes (1-e(m,m')t)^{-1}(\delta_{im}-\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j} e(m,m')t.
\end{aligned} \tag{51}$$

Hence, when $\alpha \neq \epsilon_i + \epsilon_j$, we get

$$\begin{aligned}
\Delta((D_{ij}(x^{(\alpha)}))^p) &\equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q};
\end{aligned}$$

And when $\alpha = \epsilon_i + \epsilon_j$, by Lemmas 3.4 and 4.6, (45) becomes

$$\begin{aligned}
\Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &= D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \\
&\quad + (\delta_{ik} - \delta_{jk}) h(k,k') \otimes (1 - e(k,k')t)^{-1} e(k,k')t \\
&\quad + (\delta_{im} - \delta_{jm}) h(m,m') \otimes (1 - e(m,m')t)^{-1} e(m,m')t.
\end{aligned}$$

Combining with (51), we obtain

$$\begin{aligned}
\Delta((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &\equiv ((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \otimes 1 \\
&\quad + 1 \otimes ((D_{ij}(x^{(\alpha)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q}.
\end{aligned}$$

Thereby, we prove that the ideal $I_{t,q}$ is also a coideal of the Hopf algebra $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

(II) By Lemmas 4.5, 3.2, 3.4 & 4.6, we have

$$
\begin{aligned}
S((D_{ij}(x^{(\alpha)}))^p) &= -\big(1-e(k,k')t\big)^{-p\,\alpha(k,k')}\big(1-e(m,m')t\big)^{-p\,\alpha(m,m')} \\
&\quad\cdot\Big(\sum_{n,\ell=0}^{\infty} d_{m'}m^{(n)}d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h(k,k')_1^{\langle \ell\rangle}h(m,m')_1^{\langle n\rangle}t^{n+\ell}\Big) \\
&\equiv -\sum_{n,\ell=0}^{p-1} d_{mm'}^{(n)}d_{kk'}^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h(k,k')_1^{\langle \ell\rangle}h(m,m')_1^{\langle n\rangle}t^{n+\ell} \pmod p \qquad (52)\\
&= -(D_{ij}(x^{(\alpha)}))^p+(\delta_{ik}-\delta_{jk})\delta_{\alpha,\epsilon_i+\epsilon_j}e(k,k')\cdot h(k,k')_1^{\langle 1\rangle}t \\
&\quad+(\delta_{im}-\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j}e(m,m')\cdot h(m,m')_1^{\langle 1\rangle}t.
\end{aligned}
$$

Hence, when $\alpha\neq\epsilon_i+\epsilon_j$, we get

$$S\big((D_{ij}(x^{(\alpha)}))^p\big)=-(D_{ij}(x^{(\alpha)}))^p\in I_{t,q};$$

When $\alpha=\epsilon_i+\epsilon_j$, by Lemmas 3.4 and 4.5, (46) reads as $S(D_{ij}(x^{(\epsilon_i+\epsilon_j)}))=-D_{ij}(x^{(\epsilon_i+\epsilon_j)})+(\delta_{ik}-\delta_{jk})e(k,k')\cdot h(k,k')_1^{\langle 1\rangle}t+(\delta_{im}-\delta_{jm})e(m,m')\cdot h(m,m')_1^{\langle 1\rangle}t$. Combining with (52), we obtain

$$S\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p-D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big)=-\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p-D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big)\in I_{t,q}.$$

Thereby, we show that the ideal $I_{t,q}$ is indeed preserved by the antipode S of the quantization $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

(III) It is obvious to notice that $\varepsilon((D_{ij}(x^{(\alpha)}))^p)=0$ for all $0\leq\alpha\leq\tau$.

This completes the proof. □

REMARK 4.8. Corollary 4.2 gives more Drinfel'd twists. Using the same proof as Theorem 4.7, we can get new families of noncommutative and noncocommutative Hopf algebras of dimension $p^{1+(n-1)(p^n-1)}$ in characteristic p. Obviously, they are p-polynomial deformations $(\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})),m,\iota,\ \Delta,S,\varepsilon)$ of the restricted universal enveloping algebra of $\mathbf{S}(n;\underline{1})$ over the p-truncated polynomial ring $\mathcal{K}[t]_p^{(q)}$.

4.3. Different twisted structures. We shall show that the twisted structures given by Drinfel'd twists with different product-length are nonisomorphic.

DEFINITION 4.9. A Drinfel'd twist $\mathcal{F}\in A\otimes A$ on any Hopf algebra A is called *compatible* if $\mathcal{F}$ commutes with the coproduct Δ_0.

In other words, twisting a Hopf algebra A with a *compatible* twist $\mathcal{F}$ gives exactly the same Hopf structure, that is, $\Delta_{\mathcal{F}}=\Delta_0$. The set of *compatible* twists on A thus forms a group.

LEMMA 4.10. ([**10**]) *Let $\mathcal{F}\in A\otimes A$ be a Drinfel'd twist on a Hopf algebra A. Then the twisted structure induced by $\mathcal{F}$ coincides with the structure on A if and only if $\mathcal{F}$ is a compatible twist.*

Using the same proof as in Theorem 4.1, we obtain

LEMMA 4.11. *Let $\mathcal{F},\mathcal{G}\in A\otimes A$ be Drinfel'd twists on a Hopf algebra A with $\mathcal{F}\mathcal{G}=\mathcal{G}\mathcal{F}$ and $\mathcal{F}\neq\mathcal{G}$. Then $\mathcal{F}\mathcal{G}$ is a Drinfel'd twist. Furthermore, $\mathcal{G}$ is a Drinfel'd twist on $A_{\mathcal{F}}$, $\mathcal{F}$ is a Drinfel'd twist on $A_{\mathcal{G}}$ and $\Delta_{\mathcal{F}\mathcal{G}}=(\Delta_{\mathcal{F}})_{\mathcal{G}}=(\Delta_{\mathcal{G}})_{\mathcal{F}}$.*

Let A denote one of objects: $U(\mathbf{S}_{\mathbb{Z}}^{+})[[t]]$, $U_{t,q}(\mathbf{S}(n,\underline{1}))$ and $\mathbf{u}_{t,q}(\mathbf{S}(n,\underline{1}))$.

PROPOSITION 4.12. *Drinfel'd twists* $\mathcal{F}^{\zeta(i)} := \mathcal{F}(2,1)^{\zeta_1}\cdots\mathcal{F}(n,1)^{\zeta_{n-1}}$ *(where* $\zeta(i) = (\zeta_1,\cdots,\zeta_{n-1}) = (\underbrace{1,\cdots,1}_{i},0,\cdots,0) \in \mathbb{Z}_2^{n-1}$*) lead to* $n-1$ *different twisted Hopf algebra structures on* A.

PROOF. For $i=1$, $\mathcal{F}(2,1)$ gives one twisted structure with a twisted coproduct different from the original one. For $i=2$, using Lemma 4.11, we know that $\mathcal{F}(3,1)$ is a Drinfel'd twist and not a compatible twist on $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]_{\mathcal{F}(2,1)}$. So the twist $\mathcal{F}(2,1)\mathcal{F}(3,1)$ gives new Hopf algebra structure with the coproduct different from the previous one twisted by $\mathcal{F}(2,1)$. Using the same discussion, we obtain that the Drinfel'd twists $\mathcal{F}^{\zeta(i)}$ for $\zeta(i) = (\underbrace{1,\cdots,1}_{i},0,\cdots,0) \in \mathbb{Z}_2^{n-1}$ give $n-1$ different twisted structures on $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$. This leads to the corresponding result on A. $\square$

5. Quantizations of horizontal type for $\mathbf{S}(n;\underline{1})$ and $\mathfrak{sl}_n$

In this section, we assume that $n \geq 3$. Take $h := \partial_k - \partial_{k'}$ and $e := x^{\epsilon_k-\epsilon_m}\partial_m$ ($1 \leq k \neq k' \neq m \leq n$) and denote by $\mathcal{F}(k,k';m)$ the corresponding Drinfel'd twist. These twists will lead to the quantizations in horizontal direction. So we call them the Drinfeld twists in *horizontal* (while those twists used in Sections 3, 4 are in *vertical*). Using the horizontal Drinfeld twists and the same discussion in Sections 2, 3, we obtain some new quantizations of horizontal type for the universal enveloping algebra of the special algebra $\mathbf{S}(n;\underline{1})$. The twisted structures given by the twists $\mathcal{F}(k,k';m)$ on subalgebra $\mathbf{S}(n;\underline{1})_0$ are the same as those on the special linear Lie algebra $\mathfrak{sl}_n$ over a field $\mathcal{K}$ with $\mathrm{char}(\mathcal{K}) = p$ derived by the Jordanian twists $\mathcal{F} = \exp(h\otimes\sigma)$, $\sigma = \ln(1-e)$ for some two-dimensional carrier subalgebra $B(2) = \mathrm{Span}_{\mathcal{K}}\{h,e\}$ discussed in [**13**], [**14**], etc.

5.1. Quantizations of horizontal type of $\mathbf{u}(\mathbf{S}(n;\underline{1}))$. From Lemma 2.2 and Theorem 2.4, we have

LEMMA 5.1. *Fix two distinguished elements* $h := \partial_k - \partial_{k'}$, $e := x^{\epsilon_k-\epsilon_m}\partial_m$ *($1 \leq k \neq k' \neq m \leq n$), the corresponding horizontal quantization of* $U(\mathbf{S}_{\mathbb{Z}}^+)$ *over* $U(\mathbf{S}_{\mathbb{Z}}^+)[[t]]$ *by Drinfel'd twist* $\mathcal{F}(k,k';m)$ *with the product undeformed is given by*

$$\Delta(x^\alpha\partial) = x^\alpha\partial\otimes(1-et)^{\alpha_k-\alpha_{k'}} + \sum_{\ell=0}^{\infty}(-1)^\ell\, h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot \cdot\, x^{\alpha+\ell(\epsilon_k-\epsilon_m)}(A_\ell\partial - B_\ell\partial_m)t^\ell, \tag{53}$$

$$S(x^\alpha\partial) = -(1-et)^{-(\alpha_k-\alpha_{k'})}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell(\epsilon_k-\epsilon_m)}(A_\ell\partial - B_\ell\partial_m)\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{54}$$

$$\varepsilon(x^\alpha\partial) = 0, \tag{55}$$

where $\alpha-\eta\in\mathbb{Z}_+^n$, $\eta = -\underline{1}$, $A_\ell = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}(\alpha_m - j)$, $B_\ell = \partial(\epsilon_k-\epsilon_m)A_{\ell-1}$, *with a convention* $A_0 = 1, A_{-1} = 0$.

Note that $A_\ell = 0$ for $\ell > \alpha_m$ and $B_\ell = 0$ for $\ell > \alpha_m + 1$ in Lemma 5.1.

REMARK 5.2. According to the parametrization of the twists $\mathcal{F}(k,k';m)$, we get $n(n-1)(n-2)$ *basic Drinfel'd twists* over $U(\mathbf{S}_{\mathbb{Z}}^+)$ and consider the products of some *basic Drinfel'd twists*. Using the same argument as Section 4, one can get

many more new Drinfel'd twists, which will lead to new complicated quantizations not only over the $U(\mathbf{S}^+_{\mathbb{Z}})[[t]]$, but over the $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ as well.

We firstly make *the modulo p reduction* for the quantizations of $U(\mathbf{S}^+_{\mathbb{Z}})$ in Lemma 5.1 to yield the horizontal quantizations of $U(\mathbf{S}(n;\underline{1}))$ over $U_t(\mathbf{S}(n;\underline{1}))$.

THEOREM 5.3. *Fix distinguished elements $h = D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e = D_{mk}(x^{(2\epsilon_k)})$ $(1 \le k \ne k' \ne m \le n)$, the corresponding horizontal quantization of $U(\mathbf{S}(n;\underline{1}))$ over $U_t(\mathbf{S}(n;\underline{1}))$ with the product undeformed is given by*

$$\begin{aligned}(56)\qquad \Delta(D_{ij}(x^{(\alpha)})) = D_{ij}(x^{(\alpha)}) \otimes (1-et)^{\alpha(k,k')} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{\langle \ell\rangle} \otimes (1-et)^{-\ell} \cdot \\ \cdot \Big(\bar{A}_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) + \bar{B}_\ell(\delta_{ik}D_{jm} - \delta_{jk}D_{im})(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))})\Big) t^\ell,\end{aligned}$$

$$\begin{aligned}(57)\qquad S(D_{ij}(x^{(\alpha)})) = -(1-et)^{-\alpha(k,k')} \cdot \sum_{\ell=0}^{p-1} \Big(\bar{A}_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) \\ + \bar{B}_\ell(\delta_{ik}D_{jm} - \delta_{jk}D_{im})(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))})\Big) \cdot h_1^{\langle \ell\rangle} t^\ell,\end{aligned}$$

$$(58)\qquad \varepsilon(D_{ij}(x^{(\alpha)})) = 0,$$

where $0 \le \alpha \le \tau$, $\alpha(k,k') = \alpha_k - \delta_{ik} - \delta_{jk} - \alpha_{k'} + \delta_{ik'} + \delta_{jk'}$, $\bar{A}_\ell \equiv \binom{\alpha_k+\ell}{\ell} \pmod p$ for $0 \le \ell \le \alpha_m$, $\bar{B}_\ell \equiv \binom{\alpha_k+\ell-1}{\ell-1} \pmod p$ for $1 \le \ell \le \alpha_m+1$, and otherwise, $\bar{A}_\ell = \bar{B}_\ell = 0$.

PROOF. Note that the elements $\sum_{i,\alpha} \frac{1}{\alpha!} a_{i,\alpha} x^\alpha D_i$ in $\mathbf{S}^+_{\mathcal{K}}$ for $0 \le \alpha \le \tau$ will be identified with $\sum_{i,\alpha} a_{i,\alpha} x^{(\alpha)} D_i$ in $\mathbf{S}(n;\underline{1})$ and those in $J_{\underline{1}}$ (given in Section 3.1) with 0. Hence, by Lemma 5.1, we get

$$\begin{aligned}\Delta(D_{ij}(x^{(\alpha)})) &= \frac{1}{\alpha!}\Delta(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i - \alpha_i\partial_j)) \\ &= D_{ij}(x^{(\alpha)}) \otimes (1-et)^{\alpha(k,k')} + \sum_{\ell=0}^{p-1} (-1)^\ell\, h^{\langle \ell\rangle} \otimes (1-et)^{-\ell} \cdot \\ &\quad \cdot \frac{1}{\alpha!} x^{\alpha-\epsilon_i-\epsilon_j+\ell(\epsilon_k-\epsilon_m)} \big(A_\ell(\alpha_j\partial_i - \alpha_i\partial_j) - B_\ell\partial_m\big) t^\ell,\end{aligned}$$

where $A_\ell = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}(\alpha_m - \delta_{im} - \delta_{jm} - j)$, $B_\ell = (\alpha_j\partial_i - \alpha_i\partial_j)(\epsilon_k - \epsilon_m)A_{\ell-1}$.

Write

$$\begin{aligned}(*) &= \frac{1}{\alpha!} x^{\alpha-\epsilon_i-\epsilon_j+\ell(\epsilon_k-\epsilon_m)} \big(A_\ell(\alpha_j\partial_i - \alpha_i\partial_j) - B_\ell\partial_m\big), \\ (**) &= \bar{A}_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) + \bar{B}_\ell(\delta_{ik}D_{jm} - \delta_{jk}D_{im})(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}).\end{aligned}$$

We claim that $(*) = (**)$.

The proof will be given in the following steps:

(i) For $\delta_{im} + \delta_{jm} = 1$, we have

$$(*) = \begin{cases} \frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}(A_\ell + A_{\ell-1})D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}), & \textit{for } 0 \le \ell \le \alpha_m, \\ 0, & \textit{for } \quad \ell > \alpha_m. \end{cases}$$

A simple calculation shows that $\frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}(A_\ell+A_{\ell-1}) \equiv \binom{\alpha_k+\ell}{\ell}$ (mod p), for $0 \le \ell \le \alpha_m$. So, $(*) = (**)$.

(ii) For $\delta_{im}+\delta_{jm}=0$, we consider three subcases:

If $\delta_{ik}=1$, we have

$$(*) = \begin{cases} \frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell D_{kj}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) \\ \quad + \frac{(\alpha_k+\ell-1)!}{\alpha_k!}\frac{(\alpha_m-(\ell-1))!}{\alpha_m!}A_{\ell-1}D_{jm}(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}), & \text{for } 0 \le \ell \le \alpha_m{+}1, \\ 0, & \text{for } \ell > \alpha_m{+}1. \end{cases}$$

A simple calculation indicates that for $0 \le \ell \le \alpha_m{+}1$,

$$\frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell \equiv \binom{\alpha_k+\ell}{\ell} = \bar{A}_\ell \pmod p,$$
$$\frac{(\alpha_k+\ell-1)!}{\alpha_k!}\frac{(\alpha_m-(\ell-1))!}{\alpha_m!}A_{\ell-1} \equiv \binom{\alpha_k+\ell-1}{\ell-1} = \bar{B}_\ell \pmod p.$$

So, $(*) = (**)$.

If $\delta_{jk}=1$, we have

$$(*) = \begin{cases} \frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell D_{ik}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))}) \\ \quad - \frac{(\alpha_k+\ell-1)!}{\alpha_k!}\frac{(\alpha_m-(\ell-1))!}{\alpha_m!}A_{\ell-1}D_{im}(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}), & \text{for } 0 \le \ell \le \alpha_m{+}1, \\ 0, & \text{for } \ell > \alpha_m{+}1. \end{cases}$$

A simple computation shows that

$$\frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell \equiv \binom{\alpha_k+\ell}{\ell} = \bar{A}_\ell \pmod p, \quad \text{for } 0 \le \ell \le \alpha_m,$$
$$\frac{(\alpha_k+\ell-1)!}{\alpha_k!}\frac{(\alpha_m-(\ell-1))!}{\alpha_m!}A_{\ell-1} \equiv \binom{\alpha_k+\ell-1}{\ell-1} = \bar{B}_\ell \pmod p, \quad \text{for } 0 \le \ell \le \alpha_m{+}1.$$

So, $(*) = (**)$.

If $\delta_{ik}=\delta_{jk}=0$, we have $(*) = \frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))})$, and

$$\frac{(\alpha_k+\ell)!}{\alpha_k!}\frac{(\alpha_m-\ell)!}{\alpha_m!}A_\ell \equiv \binom{\alpha_k+\ell}{\ell} = \bar{A}_\ell \pmod p, \quad \text{for } 0 \le \ell \le \alpha_m,$$
$$\bar{B}_\ell \equiv 0 \pmod p, \quad \text{for } 0 \le \ell \le \alpha_m{+}1.$$

So, $(*) = (**)$.

Therefore, we verify the formula (56).

Applying a similar argument to the antipode, we can get the formula (57).

This completes the proof. □

To describe $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$ explicitly, we still need an auxiliary Lemma.

Lemma 5.4. *Denote* $e = D_{mk}(x^{(2\epsilon_k)})$, $d^{(\ell)} = \frac{1}{\ell!}(\operatorname{ad} e)^\ell$. *Then*

(i) $d^{(\ell)}(D_{ij}(x^{(\alpha)})) = \bar{A}_\ell D_{ij}(x^{(\alpha+\ell(\epsilon_k-\epsilon_m))})$
$$+ \bar{B}_\ell(\delta_{ik}D_{jm}-\delta_{jk}D_{im})(x^{(\alpha+(\ell-1)(\epsilon_k-\epsilon_m))}),$$
where $\bar{A}_\ell, \bar{B}_\ell$ *as in Theorem 5.3.*

(ii) $d^{(\ell)}(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = \delta_{\ell,0}D_{ij}(x^{(\epsilon_i+\epsilon_j)}) - \delta_{1,\ell}(\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})e$.

(iii) $d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p) = \delta_{\ell,0}(D_{ij}(x^{(\alpha)}))^p - \delta_{1,\ell}(\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j}e$.

PROOF. We can get (i) from the proof of Theorem 5.3.

(ii) Note that $\bar{A}_0=1$, $\bar{B}_0=0$. Using Theorem 5.3, for $\delta_{im}+\delta_{jm}=1$, we obtain $\bar{A}_1=1$ and $\bar{B}_1=0$; for $\delta_{im}+\delta_{jm}=0$, we obtain $\bar{A}_1=0$ and $\bar{B}_1=1$. We have $\bar{A}_\ell=\bar{B}_\ell=0$ for $\ell>1$. Therefore, in any case, we arrive at the result as desired.

(iii) From (15), we obtain that for $0\le\alpha\le\tau$,

$$\begin{aligned}
d^{(1)}\left((D_{ij}(x^{(\alpha)}))^p\right)&=\frac{1}{(\alpha!)^p}\big[\,e,(D_{ij}(x^{\alpha}))^p\,\big]=\frac{1}{(\alpha!)^p}\big[\,e,(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i-\alpha_i\partial_j))^p\,\big]\\
&=\frac{1}{(\alpha!)^p}\sum_{\ell=1}^{p}(-1)^\ell\binom{p}{\ell}(x^{\alpha-\epsilon_i-\epsilon_j}(\alpha_j\partial_i-\alpha_i\partial_j))^{p-\ell}\cdot\\
&\qquad\cdot x^{\epsilon_k-\epsilon_m+\ell(\alpha-\epsilon_i-\epsilon_j)}(a_\ell\partial_m-b_\ell(\alpha_j\partial_i-\alpha_i\partial_j))\\
&\equiv-\frac{a_p}{\alpha!}\,x^{\epsilon_k-\epsilon_m+p(\alpha-\epsilon_i-\epsilon_j)}\partial_m\qquad(\mathrm{mod}\ \ p\,)\\
&\equiv\begin{cases}-a_p\,e, & \text{if}\quad \alpha=\epsilon_i+\epsilon_j\\ 0, & \text{if}\quad \alpha\ne\epsilon_i+\epsilon_j\end{cases}\qquad(\mathrm{mod}\ \ J),
\end{aligned}$$

where the last "$\equiv$" by using the identification with respect to modulo the ideal J as before, and $a_\ell=\prod_{m=0}^{\ell-1}(\alpha_j\partial_i-\alpha_i\partial_j)(\epsilon_k-\epsilon_m+m(\alpha-\epsilon_i-\epsilon_j))$, $b_\ell=\ell\,\partial_m(\alpha-\epsilon_i-\epsilon_j)a_{\ell-1}$, and $a_p=\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm}$, for $\alpha=\epsilon_i+\epsilon_j$.

Consequently, by the definition of $d^{(\ell)}$, we get $d^{(\ell)}((x^{(\alpha)}D_i)^p)=0$ in $\mathbf{u}(\mathbf{S}(n;\underline{1}))$ for $2\le\ell\le p-1$ and $0\le\alpha\le\tau$. $\square$

Based on Theorem 5.3 and Lemma 5.4, we arrive at

THEOREM 5.5. *Fix distinguished elements* $h=D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e=D_{mk}(x^{(2\epsilon_k)})$ $(1\le k\ne k'\ne m\le n)$, *there exists a noncommutative and noncocummtative Hopf algebra (of horizontal type)* $(\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})),m,\iota,\Delta,S,\varepsilon)$ *over* $\mathcal{K}[t]_p^{(q)}$ *with the product undeformed, whose coalgebra structure is given by*

$$\begin{aligned}
\Delta(D_{ij}(x^{(\alpha)}))&=D_{ij}(x^{(\alpha)})\otimes(1-et)^{\alpha(k,k')}\\
&\qquad+\sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}d^{(\ell)}(D_{ij}(x^{(\alpha)}))t^\ell,
\end{aligned}\tag{59}$$

$$S(D_{ij}(x^{(\alpha)}))=-(1-et)^{-\alpha(k,k')}\sum_{\ell=0}^{p-1}d^{(\ell)}(D_{ij}(x^{(\alpha)}))\cdot h_1^{\langle\ell\rangle}t^\ell,\tag{60}$$

$$\varepsilon(D_{ij}(x^{(\alpha)}))=0,\tag{61}$$

where $0\le\alpha\le\tau$ *and* $\alpha(k,k')=\alpha_k-\delta_{ik}-\delta_{jk}-\alpha_{k'}+\delta_{ik'}+\delta_{jk'}$, *which is finite dimensional with* $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))=p^{1+(n-1)(p^n-1)}$.

PROOF. Utilizing the same arguments as in the proofs of Theorems 3.5 & 4.7, we shall show that the ideal $I_{t,q}$ is a Hopf ideal of the *twisted* Hopf algebra $U_{t,q}(\mathbf{S}(n;\underline{1}))$ as in Theorem 5.3. To this end, it suffices to verify that Δ and S preserve the generators in $I_{t,q}$.

(I) By Lemmas 2.5, 5.3 & 5.4 (iii), we obtain

$$\begin{aligned}
\Delta((D_{ij}(x^{(\alpha)}))^p) &= (D_{ij}(x^{(\alpha)}))^p \otimes (1-et)^{p\,(\alpha_k-\alpha_{k'})} \\
&\quad + \sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle \ell\rangle} \otimes (1-et)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \\
&\equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + \sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle \ell\rangle} \otimes (1-et)^{-\ell} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)t^\ell \pmod{p} \qquad (62)\\
&= (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \\
&\quad + h \otimes (1-et)^{-1}(\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j} et.
\end{aligned}$$

Hence, when $\alpha \neq \epsilon_i + \epsilon_j$, we get

$$\begin{aligned}
\Delta((D_{ij}(x^{(\alpha)}))^p) &\equiv (D_{ij}(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_{ij}(x^{(\alpha)}))^p \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q}.
\end{aligned}$$

When $\alpha = \epsilon_i + \epsilon_j$, by Lemma 5.4 (ii), (56) becomes

$$\begin{aligned}
\Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &= D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \\
&\quad + h \otimes (1-et)^{-1}(\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})et.
\end{aligned}$$

Combining with (62), we obtain

$$\begin{aligned}
\Delta((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})) &\equiv \big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) \otimes 1 \\
&\quad + 1 \otimes \big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{S}(n;\underline{1})) + U_{t,q}(\mathbf{S}(n;\underline{1})) \otimes I_{t,q}.
\end{aligned}$$

Thereby, we prove that $I_{t,q}$ is a coideal of the Hopf algebra $U_{t,q}(\mathbf{S}(n;\underline{1}))$.

(II) By Lemmas 2.5, 5.3 & 5.4 (iii), we have

$$\begin{aligned}
S((D_{ij}(x^{(\alpha)}))^p) &= -(1-et)^{-p(\alpha_k-\alpha_{k'})}\sum_{\ell=0}^{\infty} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h_1^{\langle \ell\rangle}t^\ell \\
&\equiv -(D_{ij}(x^{(\alpha)}))^p - \sum_{\ell=1}^{p-1} d^{(\ell)}((D_{ij}(x^{(\alpha)}))^p)\cdot h_1^{\langle \ell\rangle}t^\ell \pmod{p} \qquad (63)\\
&= -(D_{ij}(x^{(\alpha)}))^p + (\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})\delta_{\alpha,\epsilon_i+\epsilon_j} e\cdot h_1^{\langle 1\rangle}t.
\end{aligned}$$

Hence, when $\alpha \neq \epsilon_i + \epsilon_j$, we get

$$S\big((D_{ij}(x^{(\alpha)}))^p\big) = -(D_{ij}(x^{(\alpha)}))^p \in I_{t,q}.$$

When $\alpha = \epsilon_i + \epsilon_j$, by Lemma 5.4 (ii), (57) reads as

$$S(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = -D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + (\delta_{ik}-\delta_{jk}-\delta_{im}-\delta_{jm})e\cdot h_1^{\langle 1\rangle}t.$$

Combining with (63), we obtain

$$S\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) = -\big((D_{ij}(x^{(\epsilon_i+\epsilon_j)}))^p - D_{ij}(x^{(\epsilon_i+\epsilon_j)})\big) \in I_{t,q}.$$

Thereby, we show that $I_{t,q}$ is preserved by the antipode S of $U_{t,q}(\mathbf{S}(n;\underline{1}))$ as in Theorem 5.3.

(III) It is obvious to notice that $\varepsilon((D_{ij}(x^{(\alpha)}))^p) = 0$ for all $0 \leq \alpha \leq \tau$.

So, $I_{t,q}$ is a Hopf ideal in $U_{t,q}(\mathbf{S}(n;\underline{1}))$. We get a finite-dimensional horizontal quantization on $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1}))$. □

5.2. Jordanian modular quantizations of $\mathbf{u}(\mathfrak{sl}_n)$. Let $\mathbf{u}(\mathfrak{sl}_n)$ denote the restricted universal enveloping algebra of $\mathfrak{sl}_n$. Since Drinfeld twists $\mathcal{F}(k,k';m)$ of horizontal type closely act on the subalgebra $U((\mathbf{S}^+_{\mathbb{Z}})_0)[[t]]$, consequently on $\mathbf{u}_{t,q}(\mathbf{S}(n;\underline{1})_0)$, these induce the Jordanian quantizations on $\mathbf{u}_{t,q}(\mathfrak{sl}_n)$.

By Lemma 5.4 (i), we have

$$d^{(\ell)}(D_{ij}(x^{(2\epsilon_j)})) = \delta_{\ell,0}D_{ij}(x^{(2\epsilon_j)}) + \delta_{1,\ell}\big(\delta_{jm}D_{ik}(x^{(2\epsilon_k)}) - \delta_{ik}D_{mj}(x^{(2\epsilon_j)}) + \delta_{jm}\delta_{ik}D_{km}(x^{(\epsilon_k+\epsilon_m)})\big) - \delta_{2,\ell}\delta_{jm}\delta_{ik}e.$$

By Theorem 5.3, we have

Theorem 5.6. *Fix distinguished elements $h = D_{kk'}(x^{(\epsilon_k+\epsilon_{k'})})$, $e = D_{mk}(x^{(2\epsilon_k)})$ $(1 \le k \ne k' \ne m \le n)$, the corresponding Jordanian quantization of $\mathbf{u}(\mathbf{S}(n,\underline{1})_0) \cong \mathbf{u}(\mathfrak{sl}_n)$ over $\mathbf{u}_{t,q}(\mathbf{S}(n,\underline{1})_0) \cong \mathbf{u}_{t,q}(\mathfrak{sl}_n)$ with the product undeformed, whose coalgebra structure is given by*

$$\Delta(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = D_{ij}(x^{(\epsilon_i+\epsilon_j)}) \otimes 1 + 1 \otimes D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + (\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})\, h \otimes (1-et)^{-1}et, \tag{64}$$

$$\begin{aligned}\Delta(D_{ij}(x^{(2\epsilon_j)})) &= D_{ij}(x^{(2\epsilon_j)}) \otimes (1-et)^{\delta_{jk}-\delta_{ik}-\delta_{jk'}+\delta_{ik'}} + 1 \otimes D_{ij}(x^{(2\epsilon_j)}) \\ &- h \otimes (1-et)^{-1}\big(\delta_{jm}D_{ik}(x^{(2\epsilon_k)}) - \delta_{ik}D_{mj}(x^{(2\epsilon_j)}) + \delta_{jm}\delta_{ik}D_{km}(x^{(\epsilon_k+\epsilon_m)})\big)t \\ &- \delta_{jm}\delta_{ik}\, h^{\langle 2\rangle} \otimes (1-et)^{-2}et^2,\end{aligned} \tag{65}$$

$$S(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = -D_{ij}(x^{(\epsilon_i+\epsilon_j)}) + (\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})eh_1t, \tag{66}$$

$$\begin{aligned}S(D_{ij}(x^{(2\epsilon_j)})) &= -(1-et)^{-(\delta_{jk}-\delta_{ik}-\delta_{jk'}+\delta_{ik'})} \cdot \Big(D_{ij}(x^{(2\epsilon_j)}) + \\ &\big(\delta_{jm}D_{ik}(x^{(2\epsilon_k)}) - \delta_{ik}D_{mj}(x^{(2\epsilon_j)}) + \delta_{jm}\delta_{ik}D_{km}(x^{(\epsilon_k+\epsilon_m)})\big)h_1t - \delta_{jm}\delta_{ik}eh_1^{\langle 2\rangle}t^2\Big),\end{aligned} \tag{67}$$

$$\varepsilon(D_{ij}(x^{(\epsilon_i+\epsilon_j)})) = \varepsilon(D_{ij}(x^{(2\epsilon_j)})) = 0. \tag{68}$$

for $1 \le i \ne j \le n$.

Remark 5.7. Since $\mathbf{S}(n,\underline{1})_0 \cong \mathfrak{sl}_n$, which, via the identification $D_{ij}(x^{(\epsilon_i+\epsilon_j)})$ with $E_{ii}-E_{jj}$ and $D_{ij}(x^{(2\epsilon_j)})$ with E_{ji} for $1 \le i \ne j \le n$, we get a Jordanian quantization for $\mathfrak{sl}_n$, which has been discussed by Kulish et al (cf. [**13**], [**14**] etc.).

Corollary 5.8. Fix distinguished elements $h = E_{kk} - E_{k'k'}$, $e = E_{km}$ $(1 \le k \ne k' \ne m \le n)$, the corresponding Jordanian quantization of $\mathbf{u}(\mathfrak{sl}_n)$ over $\mathbf{u}_{t,q}(\mathfrak{sl}_n)$ with the product undeformed, whose coalgebra structure is given by

$$\Delta(E_{ii}-E_{jj}) = (E_{ii}-E_{jj}) \otimes 1 + 1 \otimes (E_{ii}-E_{jj}) + (\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})h \otimes (1-et)^{-1}et, \tag{69}$$

$$\Delta(E_{ji}) = E_{ji} \otimes (1-et)^{\delta_{jk}-\delta_{ik}-\delta_{jk'}+\delta_{ik'}} + 1 \otimes E_{ji} - h \otimes (1-et)^{-1}\big(\delta_{jm}E_{ki} - \delta_{ik}E_{jm}\big)t - \delta_{jm}\delta_{ik}h^{\langle 2\rangle} \otimes (1-et)^{-2}et^2, \tag{70}$$

$$S(E_{ii}-E_{jj}) = -(E_{ii}-E_{jj}) + (\delta_{ik}-\delta_{jk}-\delta_{im}+\delta_{jm})eh_1t, \tag{71}$$

$$S(E_{ji}) = -(1-et)^{-(\delta_{jk}-\delta_{ik}-\delta_{jk'}+\delta_{ik'})}\Big(E_{ji} + \big(\delta_{jm}E_{ki} - \delta_{ik}E_{jm}\big)h_1t - \delta_{jm}\delta_{ik}eh_1^{\langle 2\rangle}t^2\Big), \tag{72}$$

$$\varepsilon(E_{ii} - E_{jj}) = \varepsilon(E_{ji}) = 0. \tag{73}$$

for $1 \leq i \neq j \leq n$.

Example 5.9. For $n = 3$, take $h = E_{11}-E_{22}$, $e = E_{13}$, and set $h' = E_{22}-E_{33}$, $f = (1-et)^{-1}$. By Corollary 5.8, we get a Jordanian quantization on $\mathbf{u}_{t,q}(\mathfrak{sl}_3)$ with the coproduct as follows (here we omit the antipode formulae which can be directly written down from (71) & (72)):

$$\begin{aligned}
\Delta(h) &= h\otimes f + 1\otimes h,\\
\Delta(h') &= h\otimes f + (h'-h)\otimes 1 + 1\otimes h',\\
\Delta(e) &= e\otimes f^{-1} + 1\otimes e,\\
\Delta(E_{12}) &= E_{12}\otimes f^{-2} + 1\otimes E_{12},\\
\Delta(E_{21}) &= E_{21}\otimes f^{2} + (1+h)\otimes E_{21} - h\otimes fE_{21}f^{-1},\\
\Delta(E_{31}) &= E_{31}\otimes f + (1+h)\otimes E_{31} - h\otimes fE_{31}f^{-1} + 2(f^{-1}-1)E_{31}\otimes f(f-1),\\
\Delta(E_{23}) &= E_{23}\otimes f + 1\otimes E_{23},\\
\Delta(E_{32}) &= E_{32}\otimes f^{-1} + (1+h)\otimes E_{32} - h\otimes fE_{32}f^{-1},
\end{aligned}$$

where $\{f, h\}$ satisfying the relations: $[h, f] = f^2 - f$, $h^p = h$, $f^p = 1$ generates the (finite-dimensional) Radford Hopf subalgebra (with f as a group-like element) over a field of characteristic p.

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Department of Mathematics, East China Normal University, Min Hang Campus, Dong Chuan Road 500, Shanghai 200241, PR China
E-mail address: `nhhu@math.ecnu.edu.cn`

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China
E-mail address: `xiulingwang@nankai.edu.cn`